



# Constitutive modeling of reinforced concrete and prestressed concrete structures strengthened by fiber-reinforced plastics

H.-T. Hu \*, F.-M. Lin, H.-T. Liu, Y.-F. Huang, T.-C. Pan

Department of Civil Engineering and Sustainable Environment Research Center, National Cheng Kung University, Tainan 701, Taiwan, ROC

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## ABSTRACT

A batch of constitutive models for steel reinforcing bar, prestressing tendon, concrete and fiber-reinforced plastic are proposed for the nonlinear finite element analysis of reinforced concrete structures, prestressed concrete structures, reinforced concrete structures strengthened by fiber-reinforced plastics and prestressed concrete structures strengthened by fiber-reinforced plastics. These material models have been tested against series of experimental data and good agreements have been obtained, which justifies the validity and the usefulness of the proposed nonlinear constitutive models.

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## 1. Introduction

Due to lightweight, high strength and good fatigue and corrosion properties, fiber-reinforced plastics (FRP) have been intensively used in the repair and strengthening of aerospace structures [1–4]. Though, the study of using FRP to strengthen concrete structures just started in the 1990s [5–16], the technology is currently widely used for both reinforced concrete structures [17–25] as well as prestressed concrete structures [26–33].

To study the behavior of reinforced concrete and prestressed concrete structures strengthened by FRP, the fundamental step is to understand the nonlinear behavior of the constitutive materials such as steel reinforcing bar, prestressing tendon, concrete and FRP, separately. The nonlinear behavior of concrete such as concrete cracking, tension stiffening, shear retention, concrete plasticity, yielding of reinforcing steel and yielding of prestressing tendon have been extensively studied by various researchers and numerous proper constitutive laws have been proposed [34–40]. However, in the literature, most studies of reinforced concrete structures strengthened by FRP have assumed that the behavior of FRP is linear. It is well known that unidirectional fibrous composites exhibit severe nonlinearity in their in-plane shear stress–strain relations [41]. In addition, deviation from linearity is also observed with in-plane transverse loading but the degree of nonlinearity is not comparable to that observed with the in-plane shear [41,42]. Therefore, appropriate modeling of the nonlinear behavior of FRP becomes crucial.

In this investigation, proper constitutive models are introduced to simulate the nonlinear behavior of steel reinforcing bar, prestressing tendon, concrete and FRP. In order to demonstrate the

capability and generality of these proposed material models suitable to model the behaviors of reinforced concrete and prestressed concrete structures strengthened by FRP, a series of numerical analyses are carried out by using the finite element program Abaqus [43] and are compared with the experimental results.

## 2. Material properties and constitutive models

The materials used in the analysis involve steel reinforcing bars, prestressing tendon, concrete and fiber-reinforced plastics. Reliable constitutive models applicable to steel reinforcing bars, prestressing tendon and concrete are available in the Abaqus material library. Thus, their input material properties and associated constitutive models are briefly discussed. The Abaqus program does not have a nonlinear material library for FRP. Hence, its nonlinear constitutive model is discussed here in detail. The resulting nonlinear constitutive equations for the FRP are coded in FORTRAN language as a subroutine and linked to the Abaqus program.

### 2.1. Steel reinforcing bar

The elastic modulus of the steel reinforcement used in the analyses is assumed to be

$$E_s = 200 \text{ GPa} \quad (1)$$

The stress–strain curve of the reinforcing bar is assumed to be elastic perfectly plastic.

### 2.2. Prestressing tendon

Unless specified, the elastic modulus of the prestressing tendon used in the analyses is also assumed to be  $E_s = 200 \text{ GPa}$ . Based on

\* Corresponding author. Tel.: +886 6 2757575; fax: +886 6 2358542.  
E-mail address: [hthu@mail.ncku.edu.tw](mailto:hthu@mail.ncku.edu.tw) (H.-T. Hu).

the material tested data, the nonlinear stress–strain curve of the tendon can be simplified to a piecewise linear curve and input to Abaqus.

In Abaqus, the prestressing tendon and the steel reinforcement are treated as equivalent uniaxial materials, which are smeared through the element section. In order to properly model the constitutive behaviors of the tendon and the reinforcement, the cross sectional area, spacing, position and orientation of each layer of tendon or steel bar within each element need to be specified.

### 2.3. Concrete

The Poisson's ratio  $\nu_c$  of concrete under uniaxial compressive stress ranges from 0.15 to 0.22, with a representative value of 0.19 or 0.20 [34]. In this study, the Poisson's ratio of concrete is assumed to be

$$\nu_c = 0.2 \quad (2)$$

Let the uniaxial compressive strength and the corresponding strain of the concrete be  $f'_c$  and  $\epsilon'_c$ . The value of  $\epsilon'_c$  is usually around the range of 0.002–0.003. A representative value suggested by ACI Committee 318 [44] and used in the analysis is

$$\epsilon_o = 0.003 \quad (3)$$

The uniaxial tensile strength  $f'_t$  of concrete is difficult to measure. For this study the value is taken as [34]

$$f'_t = 0.33\sqrt{f'_c} \text{ MPa} \quad (4)$$

The initial modulus of elasticity of concrete  $E_c$  is highly correlated to its compressive strength and can be calculated with reasonable accuracy from the empirical equation [44]

$$E_c = 4700\sqrt{f'_c} \text{ MPa} \quad (5)$$

Under multiaxial combinations of loading, the failure strengths of concrete are different from those observed under uniaxial condition. However, the maximum strength envelope under multiple stress conditions seems to be largely independent of load path [45]. In Abaqus, a Mohr–Coulomb type compression surface together with a crack detection surface is used to model the failure surface of concrete (Fig. 1). When the principal stress components of concrete are in biaxial compression zone, the response of the concrete is modeled by an elastic–plastic theory with an associated flow and an isotropic hardening rule. When the principal stress components of concrete are in biaxial tension zone or in biaxial tension–compression zone, cracking of concrete is defined to occur by the crack detection surface. Once cracking of concrete takes place, the orientation of the crack is stored. Damaged elasticity is then used to model the existing crack [43].

When plastic deformation occurs, there should be a certain parameter to guide the expansion of the yield surface. A commonly used approach is to relate the multidimensional stress and strain conditions to a pair of quantities, namely, the effective stress  $\sigma_c$  and effective strain  $\epsilon_c$ , such that results obtained following different loading paths can all be correlated by means of the equivalent uniaxial stress–strain curve. The stress–strain relationship proposed by Saenz [46] has been widely adopted as the uniaxial stress–strain curve for concrete and it has the following form:

$$\sigma_c = \frac{E_c \epsilon_c}{1 + (R + R_E - 2)\left(\frac{\epsilon_c}{\epsilon_o}\right) - (2R - 1)\left(\frac{\epsilon_c}{\epsilon_o}\right)^2 + R\left(\frac{\epsilon_c}{\epsilon_o}\right)^3} \quad (6)$$

where

$$R = \frac{R_E(R_\sigma - 1)}{(R_E - 1)^2} - \frac{1}{R_E}, \quad R_E = \frac{E_c}{E_o}, \quad E_o = \frac{f'_c}{\epsilon_o}$$

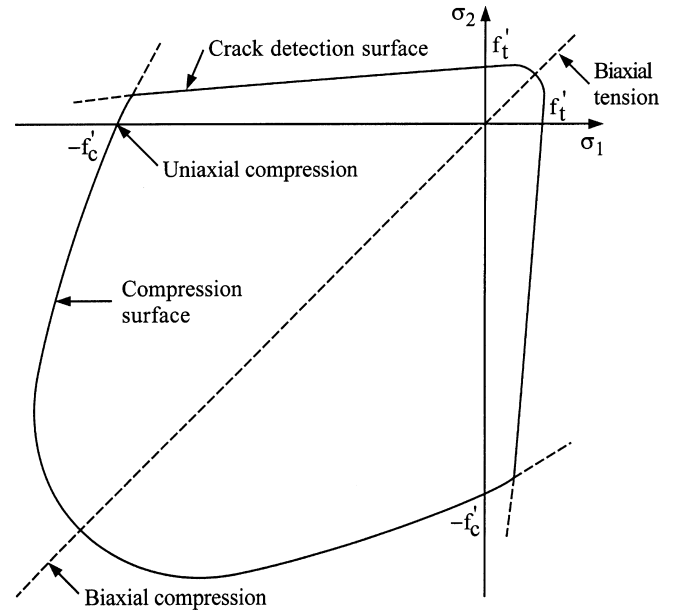


Fig. 1. Concrete failure surface in-plane stress.

and  $R_\sigma = 4$ ,  $R_\epsilon = 4$  may be used [38]. In the analysis, Eq. (6) is taken as the equivalent uniaxial stress–strain curve for concrete and approximated by several piecewise linear segments inputted to Abaqus.

When cracking of concrete takes place, a smeared model is used to represent the discontinuous macrocrack behavior. It is known that the cracked concrete of a reinforced concrete element can still carry some tensile stress in the direction normal to the crack, which is termed tension stiffening [34]. In this study, a simple descending line is used to model this tension stiffening phenomenon (Fig. 2). The default value of the strain  $\epsilon^*$  at which the tension stiffening stress reduced to zero is [43]

$$\epsilon^* = 0.001 \quad (7)$$

During the postcracking stage, the cracked reinforced concrete can still transfer shear forces through aggregate interlock or shear friction, which is termed shear retention. Assuming that the shear modulus of intact concrete is  $G_c$ , then the reduced shear modulus  $\hat{G}$  of cracked concrete can be expressed as

$$\hat{G} = \mu G_c \quad (8)$$

$$\mu = (1 - \epsilon/\epsilon_{\max}) \quad (9)$$

where  $\epsilon$  is the strain normal to the crack direction and  $\epsilon_{\max}$  is the strain at which the parameter  $\mu$  reduces to zero. Numerous analyt-

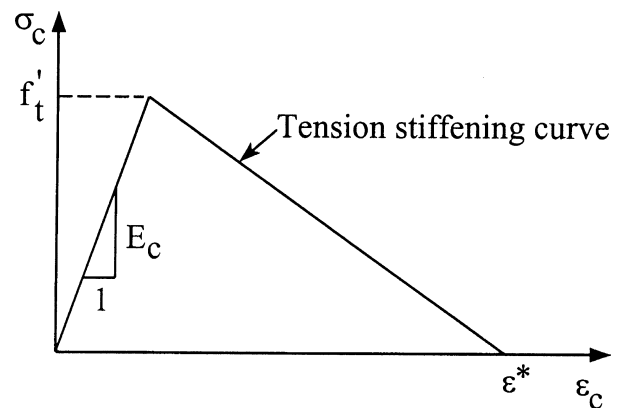


Fig. 2. Tension stiffening model.

ical results have demonstrated that the particular value chosen for  $\mu$  (between 0 and 1) does not appear to be critical but values greater than zero are necessary to prevent numerical instabilities [34,39]. In Abaqus,  $\varepsilon_{\max}$  is usually assumed to be a very large value, i.e.,  $\mu = 1$  (full shear retention). In this investigation, the default values for tension stiffening parameter  $\varepsilon^* = 0.001$  and for shear retention parameter  $\mu = 1$  are used.

2.4. Fiber-reinforced plastics

For fiber-reinforced plastics (Fig. 3), each lamina can be considered as an orthotropic layer in a plane stress condition. It is well known that unidirectional fibrous composites exhibit severe nonlinearity in their in-plane shear stress–strain relation. In addition, deviation from linearity is also observed with in-plane transverse loading but the degree of nonlinearity is not comparable to that in the in-plane shear [41]. Usually, this nonlinearity associated with the transverse loading can be ignored [42]. To model the nonlinear in-plane shear behavior, the nonlinear strain–stress relation for a composite lamina suggested by Hahn and Tsai [41] is adopted as follows:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & 0 \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} + S_{6666} \tau_{12}^2 \begin{Bmatrix} 0 \\ 0 \\ \tau_{12} \end{Bmatrix} \quad (10)$$

In this model only one constant  $S_{6666}$  is required to account for the in-plane shear nonlinearity. The value of  $S_{6666}$  can be determined by a curve fit to various off-axis tension test data [41]. Let us define  $\Delta\{\sigma'\} = \Delta\{\sigma_1, \sigma_2, \tau_{12}\}^T$  and  $\Delta\{\varepsilon'\} = \Delta\{\varepsilon_1, \varepsilon_2, \gamma_{12}\}^T$ . Inverting and differentiating Eq. (10), the incremental stress–strain relations are established.

$$\Delta\{\sigma'\} = [Q'_1] \Delta\{\varepsilon'\} \quad (11)$$

$$[Q'_1] = \begin{bmatrix} \frac{E_{11}}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} & \frac{E_{22}}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & \frac{1}{1/G_{12}+3S_{6666}\tau_{12}^2} \end{bmatrix} \quad (12)$$

Furthermore, it is assumed that the transverse shear stresses always behave linearly and do not affect the nonlinear behavior of any in-plane shear. If we define  $\Delta\{\tau'_t\} = \Delta\{\tau_{13}, \tau_{23}\}^T$  and  $\Delta\{\gamma'_t\} = \Delta\{\gamma_{13}, \gamma_{23}\}^T$ , the constitutive equations for transverse shear stresses become

$$\Delta\{\tau'_t\} = [Q'_2] \Delta\{\gamma'_t\} \quad (13)$$

$$[Q'_2] = \begin{bmatrix} \alpha_1 G_{13} & 0 \\ 0 & \alpha_2 G_{23} \end{bmatrix} \quad (14)$$

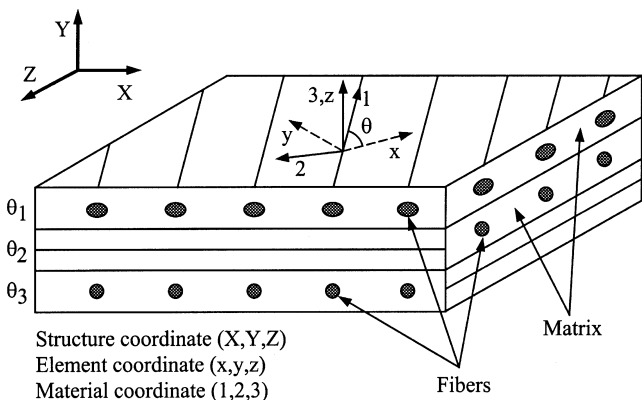


Fig. 3. Material, element and structure coordinates of fiber-reinforced plastics.

where  $\alpha_1$  and  $\alpha_2$  are the shear correction factors and are taken to be 0.83 in this study.

Among existing failure criteria, the Tsai–Wu criterion [47] has been extensively used in the literature and is adopted in this analysis. Under plane stress conditions, this failure criterion has the following form:

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 = 1 \quad (15)$$

with

$$F_1 = \frac{1}{\bar{X}} + \frac{1}{\bar{X}'}, \quad F_2 = \frac{1}{\bar{Y}} + \frac{1}{\bar{Y}'}, \quad F_{11} = \frac{-1}{\bar{X}\bar{X}'}, \quad F_{22} = \frac{-1}{\bar{Y}\bar{Y}'}, \quad F_{66} = \frac{1}{\bar{S}^2}$$

The  $\bar{X}, \bar{Y}$  and  $\bar{X}', \bar{Y}'$  are the lamina longitudinal and transverse strengths in tension and compression, respectively, and  $\bar{S}$  is the shear strength of the lamina. Though the stress interaction term  $F_{12}$  in Eq. (15) is difficult to be determined, it has been suggested that  $F_{12}$  can be set equal to zero for practical engineering applications [48]. Therefore,  $F_{12} = 0$  is used in this investigation.

During the numerical calculation, incremental loading is applied to composite plates until failures in one or more of individual plies are indicated according to Eq. (15). Since the Tsai–Wu criterion does not distinguish failure modes, the following two rules are used to determine whether the ply failure is caused by resin fracture or fiber breakage [49]:

- (1) If a ply fails but the stress in the fiber direction remains less than the uniaxial strength of the lamina in the fiber direction, i.e.  $\bar{X}' < \sigma_1 < \bar{X}$ , the ply failure is assumed to be resin induced. Consequently, the laminate loses its capability to support transverse and shear stresses, but remains to carry longitudinal stress. In this case, the constitutive matrix of the lamina becomes
- $$[Q'_1] = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (16)$$
- (2) If a ply fails with  $\sigma_1$  exceeding the uniaxial strength of the lamina, the ply failure is caused by the fiber breakage and a total ply rupture is assumed. In this case, the constitutive matrix of the lamina becomes

$$[Q'_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (17)$$

During a finite element analysis, the constitutive matrix of composite materials at the integration points of shell elements must be calculated before the stiffness matrices are assembled from the element level to the structural level. For composite materials, the incremental constitutive equations of a lamina in the element coordinates  $(x, y, z)$  can be written as:

$$\Delta\{\sigma\} = [Q_1] \Delta\{\varepsilon\} \quad (18)$$

$$\Delta\{\tau_t\} = [Q_2] \Delta\{\gamma_t\} \quad (19)$$

where  $\Delta\{\sigma\} = \Delta\{\sigma_x, \sigma_y, \tau_{xy}\}^T$ ,  $\Delta\{\tau_t\} = \Delta\{\tau_{xz}, \tau_{yz}\}^T$ ,  $\Delta\{\varepsilon\} = \Delta\{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T$ ,  $\Delta\{\gamma_t\} = \Delta\{\gamma_{xz}, \gamma_{yz}\}^T$ , and

$$[Q_1] = [T_1]^T [Q'_1] [T_1] \quad (20)$$

$$[Q_2] = [T_2]^T [Q'_2] [T_2] \quad (21)$$

$$[T_1] = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & -\sin \theta \cos \theta \\ -2 \sin \theta \cos \theta & 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \quad (22)$$

$$[T_2] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad (23)$$

The  $\theta$  is measured counterclockwise from the element local  $x$ -axis to the material 1-axis (Fig. 3). Let  $\Delta\{\varepsilon\}_o = \Delta\{\varepsilon_{x_o}, \varepsilon_{y_o}, \varepsilon_{x_{y_o}}\}^T$  be the incremental in-plane strains at the mid-surface of the shell section and  $\Delta\{\kappa\} = \Delta\{\kappa_x, \kappa_y, \kappa_{xy}\}^T$  the incremental curvatures. The incremental in-plane strains at a distance  $z$  from the mid-surface of the shell section become

$$\Delta\{\varepsilon\} = \Delta\{\varepsilon_o\} + z\Delta\{\kappa\} \quad (24)$$

Let  $h$  be the total thickness of the composite shell section, the incremental stress resultants,  $\Delta\{N\} = \Delta\{N_x, N_y, N_{xy}\}^T$ ,  $\Delta\{M\} = \Delta\{M_x, M_y, M_{xy}\}^T$  and  $\Delta\{V\} = \Delta\{V_x, V_y\}^T$  can be defined as:

$$\begin{Bmatrix} \Delta\{N\} \\ \Delta\{M\} \\ \Delta\{V\} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \Delta\{\sigma\} \\ z\Delta\{\sigma\} \\ \Delta\{\tau_t\} \end{Bmatrix} dz \quad (25)$$

Substituting Eqs. (18), (19), and (24) into the above expression, one can obtain the stiffness matrix for the fiber composite laminate shell at the integration point as

$$\begin{Bmatrix} \Delta\{N\} \\ \Delta\{M\} \\ \Delta\{V\} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} [Q_1] & z[Q_1] & [0] \\ z[Q_1] & z^2[Q_1] & [0] \\ [0]^T & [0]^T & [Q_2] \end{bmatrix} \begin{Bmatrix} \Delta\{\varepsilon_o\} \\ \Delta\{\kappa\} \\ \Delta\{\gamma_t\} \end{Bmatrix} dz \quad (26)$$

where  $[0]$  is a  $3 \times 2$  null matrix.

### 3. Numerical analysis

The validity of the material models for steel, FRP and reinforced concrete has been verified individually by testing against experimental data [43,50] and is not duplicated here. The validity of these material models to simulate the behavior of prestressed concrete, reinforced concrete strengthened by FRP and prestressed concrete strengthened by FRP are demonstrated in this section. In the numerical analyses, the Riks method [43,51] in Abaqus is used to carry out the nonlinear finite element solutions.

**Table 1**  
Prestress, lateral reinforcement and ultimate load for Durrani-Elias specimens.

Specimens	Prestress (MPa)	Spacing of lateral reinforcement (mm)	Ultimate load (MPa)		
			Experimental	Numerical	Error (%)
A11	930	None	36.7	36.4	0.82
A12	770	None	33.2	33.7	1.51
B2	770	51	38.6	37.9	1.81
B3	770	75	37.2	38.2	2.59

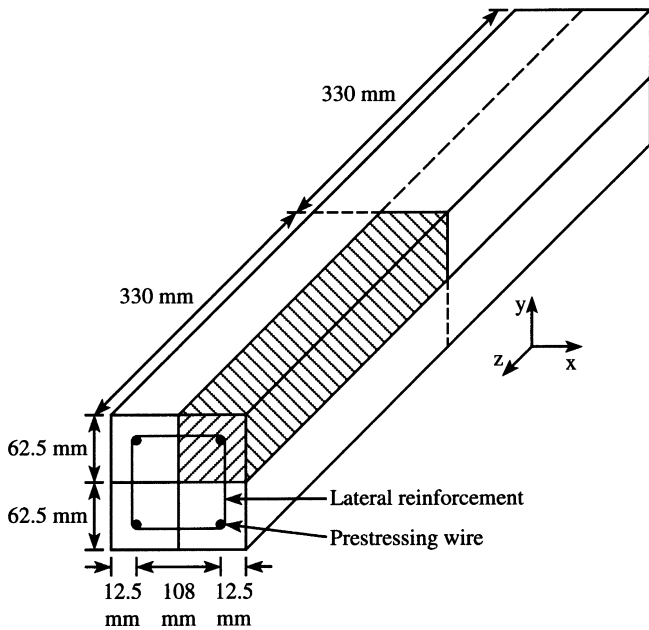


Fig. 4. Durrani-Elias specimens for prestressed concrete columns.

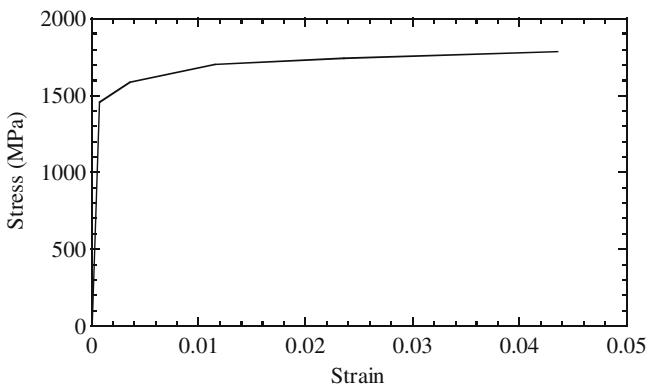


Fig. 5. Stress-strain relation of prestressing steel wire for Durrani-Elias specimens.

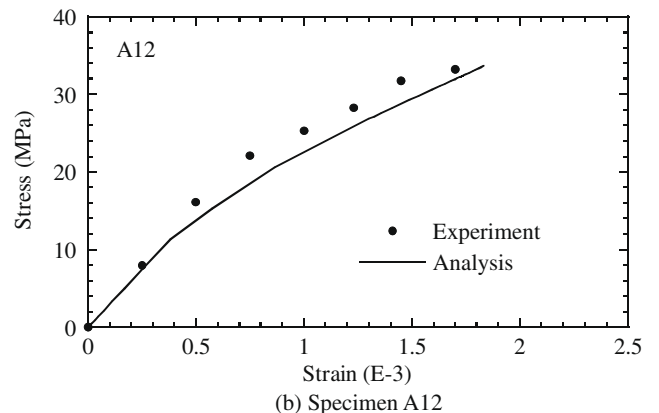
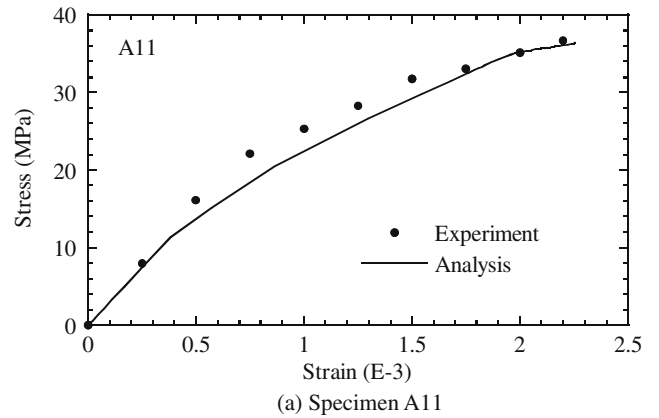


Fig. 6. Stress-strain curves of prestressed concrete columns without lateral reinforcement.

3.1. Axially loaded prestressed concrete columns

To evaluate the ability of the proposed material responses in modeling the prestressed concrete structures, the best starting point is to test them against structures that are subjected to uniform compressive forces. For this purpose, the experimental work of Durrani and Elias [52] is chosen.

The dimensions of the prestressed concrete columns are shown in Fig. 4. The uniaxial compressive strength of the concrete is  $f'_c = 38$  MPa. The columns have four prestressing steel wires and each steel wire has a diameter of 5 mm. The stress–strain relation of the wire is shown in Fig. 5. While the specimens A11 and A12 do not have lateral reinforcement, the spacing of the steel lateral reinforcement of the specimens B2 and B3 are given in Table 1. The steel lateral reinforcement has a diameter of 6 mm and a yield stress  $f_y = 620$  MPa. The prestress applied to each specimen is also given in Table 1. These specimens are fixed at one end and subjected to axial compressive force at the other end up to failure.

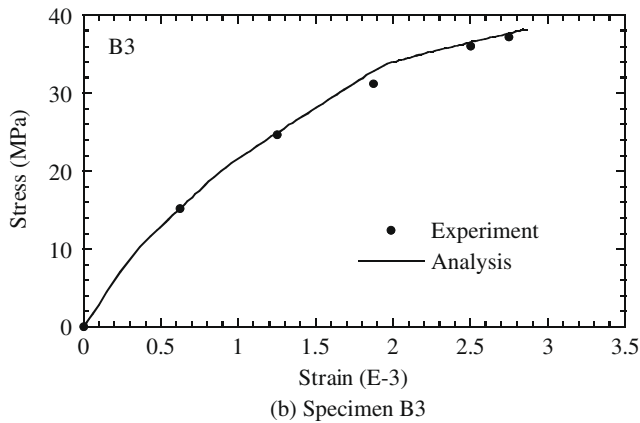
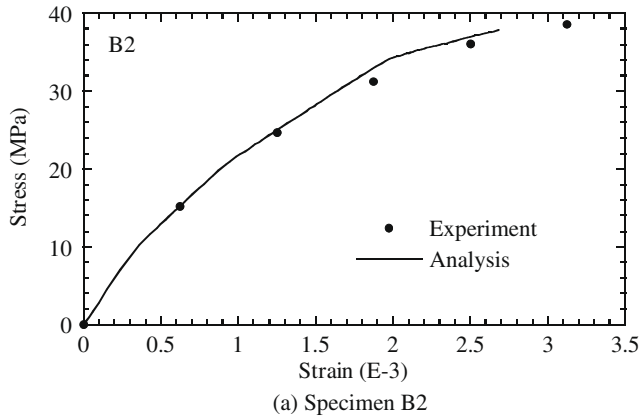


Fig. 7. Stress–strain curves of prestressed concrete columns with lateral reinforcement.

Due to symmetry, only 1/8 portion of the specimen (the shaded part in Fig. 4) is analyzed and symmetric boundary conditions are placed along the symmetric planes. Based on the result of convergence test [53], 8-node plane stress elements with reduced integration rule and with  $1 \times 10$  mesh (1 row in  $y$ -direction and 10 rows in  $z$ -direction) are used to model the prestressed concrete columns. Figs. 6 and 7 show the axial stress versus axial strain curves for all the four specimens. It can be seen that the proposed material models are satisfactory in modeling the compressive behavior of the prestressed concrete columns subjected to various prestressing forces and with various lateral reinforcement spacing. Generally, the calculated ultimate strengths are in good agreement with the experiment results (Table 1).

3.2. Laterally loaded prestressed concrete beam

In this example, a prestressed I beam tested by Rabczuk and Eibl [54] are studied. The dimensions of the beam can be found in Fig. 8.

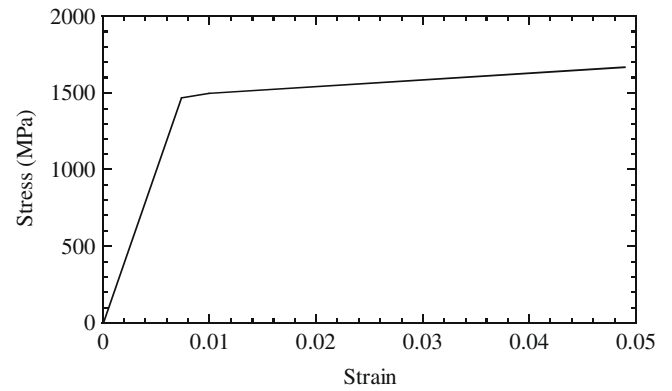


Fig. 9. Stress–strain relation of steel wire for Rabczuk–Eibl specimen.

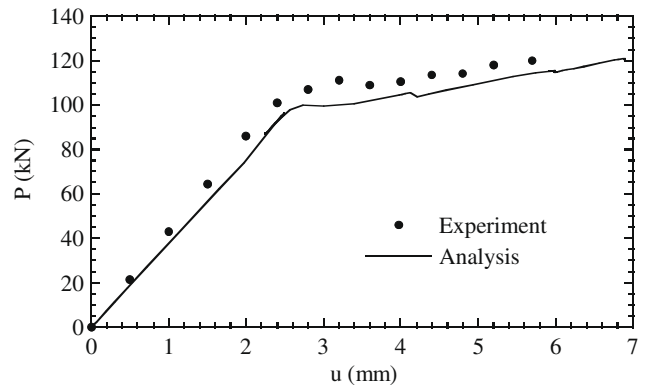


Fig. 10. Load–deflection curves of prestressed concrete beam.

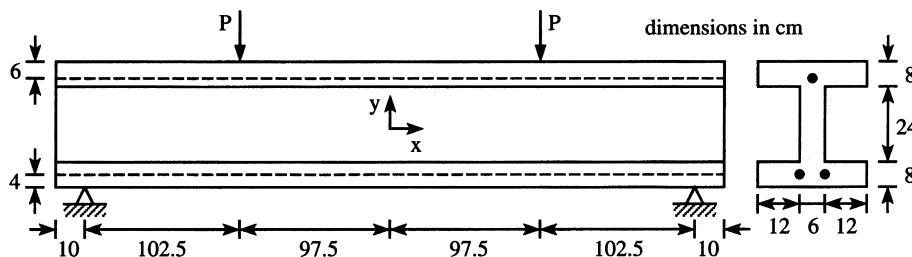


Fig. 8. Rabczuk–Eibl specimen for prestressed concrete beam.

The uniaxial compressive strength of the concrete is  $f'_c = 45$  MPa. The beam has two tension wires of 12 mm diameter at the lower flange. Each wire was prestressed with a force of 80 kN. The diameter of the upper reinforcement is 10 mm. The stress–strain relation of the wire is shown in Fig. 9. After the prestressed forces are applied, the beam is subjected to four-point static load up to failure.

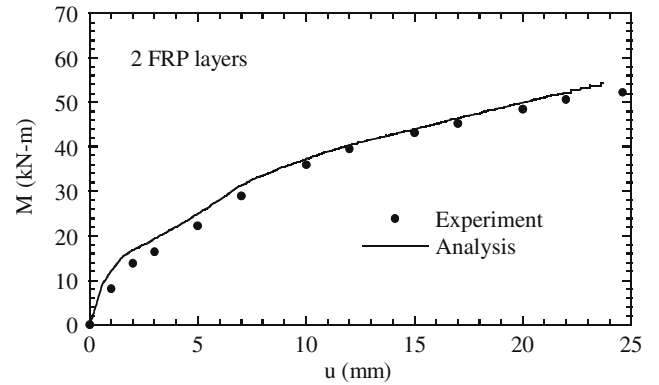
Due to symmetry, only 1/4 portion of the specimen is analyzed and symmetric boundary conditions are placed along the symmetric planes. Based on the result of convergence test [53], 8-node plane stress elements with reduced integration rule and with  $21 \times 5$  mesh (21 row in  $x$ -direction and 5 rows in  $y$ -direction) are used to model the prestressed concrete beam. Fig. 10 shows the lateral load  $P$  versus vertical displacement  $u$  at the midspan of the beam. It can be seen that the proposed material models are satisfactory in modeling the flexural behavior of the prestressed concrete beam subjected to lateral loading. The calculated ultimate load  $P_u = 121$  kN is also in good agreement with the experiment result  $P_u = 120$  kN.

3.3. Laterally loaded reinforced concrete beam strengthened by FRP

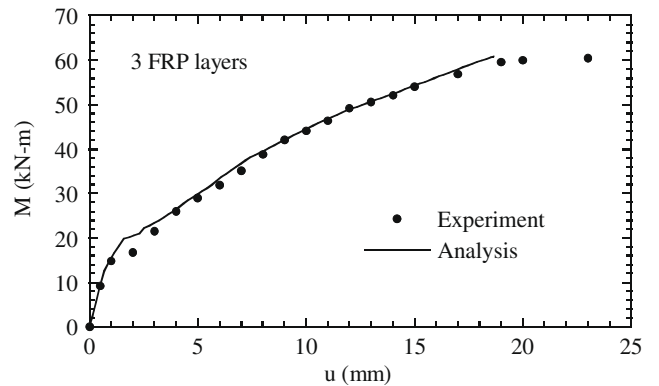
The validity of the proposed material models to simulate the composite behavior of reinforced concrete beam strengthened by FRP is examined in this section by comparing with the result of beam experiment performed by Shahawy et al. [11]. The dimensions of the test beam are given in Fig. 11. The beam is subjected to four-point static load up to failure. The flexural reinforcement is composed of two 13 mm diameter steel bars in tension zone and two 3 mm diameter steel bars in compression zone. The yielding strength  $\sigma_y$  of the reinforcing steel is 468.8 MPa and the compressive strength  $f'_c$  of concrete is 41.37 MPa.

The beams were strengthened with either 2 or 3 layers of FRP adhered to the bottom face of the beam with their fiber directions oriented in the axial direction of the beam. Each FRP layer is 0.17 mm in thickness with tensile strength  $\bar{X} = 2758$  MPa and modulus  $E_{11} = 141.3$  GPa. To take the Tsai–Wu criterion into account, the following parameters are assumed:  $\bar{X}' = \bar{Y}' = -27.58$  MPa,  $\bar{Y} = \bar{S} = 27.58$  MPa,  $E_{22} = G_{12} = 1413$  MPa,  $S_{6666} = 7.32$  GPa<sup>-3</sup>,  $\nu_{12} = 0.21$ . Since the FRP layers are subjected to uniaxial tension in fiber direction only, these assumed parameters would not affect the uniaxial tensile behavior of the FRP.

The beams have two planes of symmetry. Due to symmetry, only 1/4 portion of the beam is analyzed and symmetric boundary conditions are placed along the two symmetric planes. In the finite element analysis, 8-node solid elements with reduced integration rule are used to model the reinforced concrete beams. Based on the result of convergence test [55], the mesh of the beam has 78 solid elements in total (26 rows in  $x$ -direction, 3 rows in  $y$ -direction).

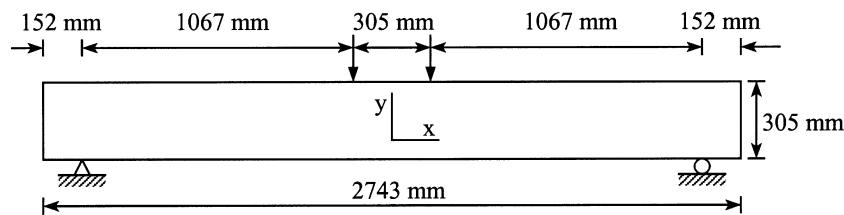


(a) Reinforced concrete beams strengthened by 2 FRP layers

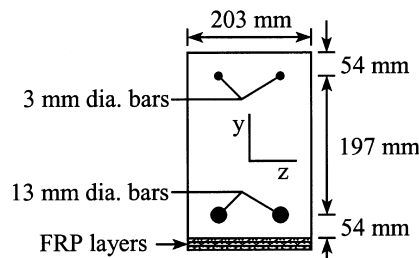


(b) Reinforced concrete beams strengthened by 3 FRP layers

Fig. 12. Moment–deflection curves of reinforced concrete beams strengthened by various FRP layers.



(a) Longitudinal section



(b) Cross section

Fig. 11. Shahawy et al. specimens for reinforced concrete beams.



tion and 1 row in z-direction). Because the fiber-reinforced plastics are relatively thin compared to the concrete beam, they are modeled by the 4-node shell elements with reduced integration rule. The FRP shell elements are attached to the bottom surface of the beam directly and perfect bonding between FRP and the concrete is assumed.

Fig. 12 shows the moment versus deflection curves of the beam at the midspan. It can be observed that the correlations are quite good between the numerical results and the experimental data. For beam strengthened by 2 FRP layers, the predicted ultimate moment 54.3 kN-m is in good agreement with the experimental ultimate moment 52.2 kN-m. The error is about 4.0%. For beam strengthened by 3 FRP layers, the predicted ultimate moment 60.9 kN-m is also in good agreement with the experimental ultimate moment 60.4 kN-m. The error is only 0.8%.

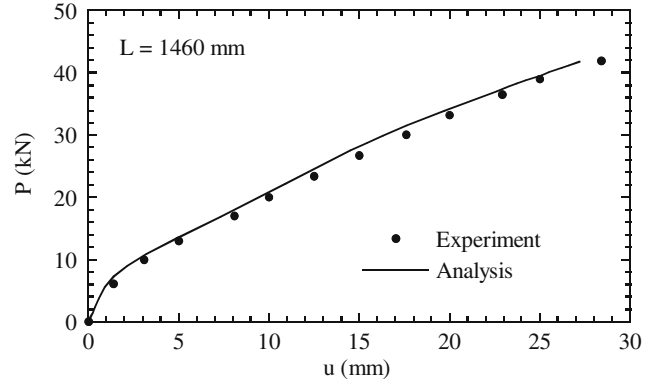
3.4. Laterally loaded reinforced concrete one-way slabs strengthened by FRP

In this example, two reinforced concrete one-way slabs strengthened by FRP and tested by Seim et al. [17] are studied. The dimensions of the slab can be found in Fig. 13. The distance from the extreme compression face to the centroid of the longitudinal tension reinforcement was 83 mm. In the longitudinal direction, the reinforcement consisted of three #3 rebars (9.5 mm diameter), whereas 23 #2 rebars (6.4 mm diameter) at 100 mm spacing were placed in the transverse direction. The yielding strength  $\sigma_y$  of the reinforcing steel is 462 MPa and the compressive strength  $f'_c$  of concrete is 33.2 MPa.

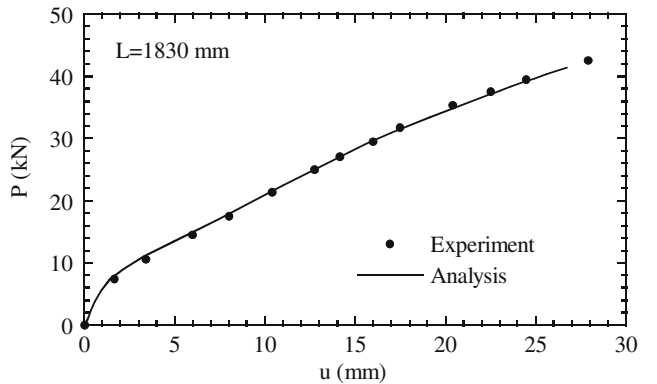
Each slab was strengthened with 2 FRP strips adhered to the bottom face of the slab with their fiber directions oriented in the longitudinal direction (Fig. 13). Two different lengths of strip, i.e.  $L = 1830$  mm (full length) or  $1460$  mm (median length), were used. Each FRP strip is 1.19 mm in thickness with tensile strength  $\bar{X} = 2270$  MPa and modulus  $E_{11} = 198$  GPa. In addition, the following parameters are assumed:  $\nu_{12} = 0.21$ ,  $\bar{X}' = \bar{Y}' = -22.7$  MPa,  $\bar{Y} = \bar{S} = 22.7$  MPa,  $E_{22} = G_{12} = 1980$  MPa,  $S_{6666} = 7.32$  GPa<sup>-3</sup>.

The slabs have two planes of symmetry. Due to symmetry, only 1/4 portion of the slab is analyzed and symmetric boundary conditions are placed along the two symmetric planes. In the finite element analysis, 8-node shell elements with reduced integration rule are used to model the reinforced concrete slabs. Based on the result

of convergence test [55], the finite element mesh has 27 reinforced concrete shell elements in total (9 rows in the longitudinal direction, 3 rows in the transverse direction). The fiber-reinforced plastics are also modeled by the 8-node shell elements with reduced integration rule and perfect bonding between FRP and the slab is



(a) Reinforced concrete one-way slab strengthened by FRP of median length



(b) Reinforced concrete one-way slab strengthened by FRP of full length

Fig. 14. Load-deflection curves of reinforced concrete one-way slabs strengthened by various lengths of FRP.

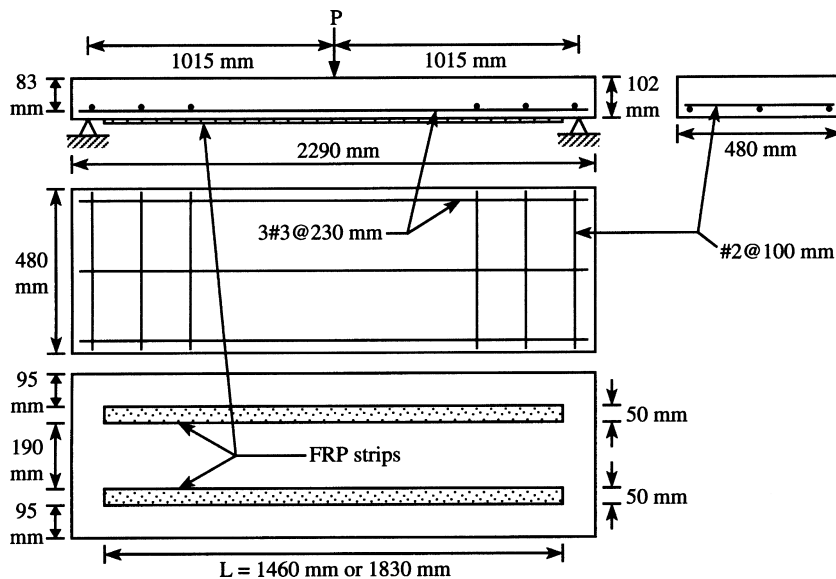


Fig. 13. Seim et al. specimens for reinforced concrete one-way slabs.

assumed. For slab with strip of full length, 7 FRP shell elements are used. For slab with strip of median length, 6 FRP shell elements are employed. In the experiment, when the slabs failed, the longitudinal strain in the FRP strips is measured and is equal to 0.0065. This maximum strain criterion, i.e.  $\epsilon_1 \leq 0.0065$ , is used together with the Tsai–Wu criterion for FRP in this numerical analysis.

Fig. 14 shows the lateral load  $P$  versus vertical displacement  $u$  at the midspan of the slabs. It can be observed that the correlation is quite good between the numerical result and the experimental data. For slab strengthened by FRP strips of median length, the predicted ultimate load 41.8 kN is in good agreement with the experimental ultimate load 41.9 kN. The error is only 0.2%. For slab strengthened by FRP strips of full length, the predicted ultimate load 41.5 kN is also in good agreement with the experimental ultimate load 42.5 kN. The error is about 2.4%. The ultimate loads of these two strengthened slabs are very close. This indicates that the use of FRP strips of full length to strengthen the slab may not be necessary.

3.5. Laterally loaded reinforced concrete two-way slabs strengthened by FRP

In this example, two square reinforced concrete two-way slabs tested by Mosallam and Mosalam [20] are studied. The clear span of the two-way slabs in both directions was 264 cm with constant thickness of 76.2 mm as shown in Fig. 15. The slabs are simply supported at the four edges and are subjected to a uniform static pressure applied to the bottom surface of the slabs up to failure. The compressive strength  $f'_c$  of concrete is 41.37 MPa. Both reinforced concrete slabs were fabricated using Grade 60 reinforcing steel with the yielding strength  $\sigma_y$  equal to 414 MPa. Tension (top) reinforcement consisted of #3 rebars (9.5 mm diameter) at 305 mm equal spacing in the two orthogonal directions of each slab with 13 mm cover. While one specimen is purely reinforced concrete

slab, the other specimen is strengthened by two FRP layers spaced at 457 mm and adhered to the top side with their fiber directions oriented in the two orthogonal directions of the slab (Fig. 15). At the intersection regions of the staggered unidirection laminates, a bidirectional fiber architecture, i.e. [90/0/90/0] lamination layup, was formed. Each FRP layer is 0.58 mm in thickness with tensile strength  $\bar{X} = 1246$  MPa and modulus  $E_{11} = 104$  GPa. In addition, the following parameters are assumed:  $\nu_{12} = 0.3$ ,  $\bar{X}' = -1246$  MPa,  $\bar{Y} = 187$  MPa,  $\bar{Y}' = -93$  MPa,  $\bar{S} = 93$  MPa,  $E_{22} = 6.75$  GPa,  $G_{12} = 7.1$  GPa,  $S_{6666} = 7.32$  GPa<sup>-3</sup>.

The slabs have two planes of symmetry. Due to symmetry, only 1/4 portion of the slab is analyzed and symmetric boundary conditions are placed along the two symmetric planes. In the finite element analysis, 8-node shell elements with reduced integration rule are used to model the reinforced concrete slabs. Based on the result of convergence test [55], the finite element mesh has 25 reinforced concrete shell elements in total (5 rows in each FRP strip direction). The fiber-reinforced plastics are also modeled by the 8-node shell elements with reduced integration rule and perfect bonding between FRP and the slab is assumed. For slab strengthened with FRP strips, 21 FRP shell elements are used.

Fig. 16a shows the total lateral force versus vertical displacement at the midspan of the reinforced concrete two-way slab not strengthened by FRP. It can be observed that the correlation is quite good between the numerical result and the experimental data. The predicted ultimate load 217 kN is in good agreement with the experimental ultimate load 219 kN with an error of 0.9%. This again verifies the validity of the proposed material models to model the behavior of reinforced concrete structures.

Fig. 16b shows the total lateral force versus vertical displacement at the midspan of the reinforced concrete two-way slab strengthened by FRP. It can be observed that the correlation is also quite good between the numerical result and the experimental data. The predicted ultimate load 394 kN is in good agreement

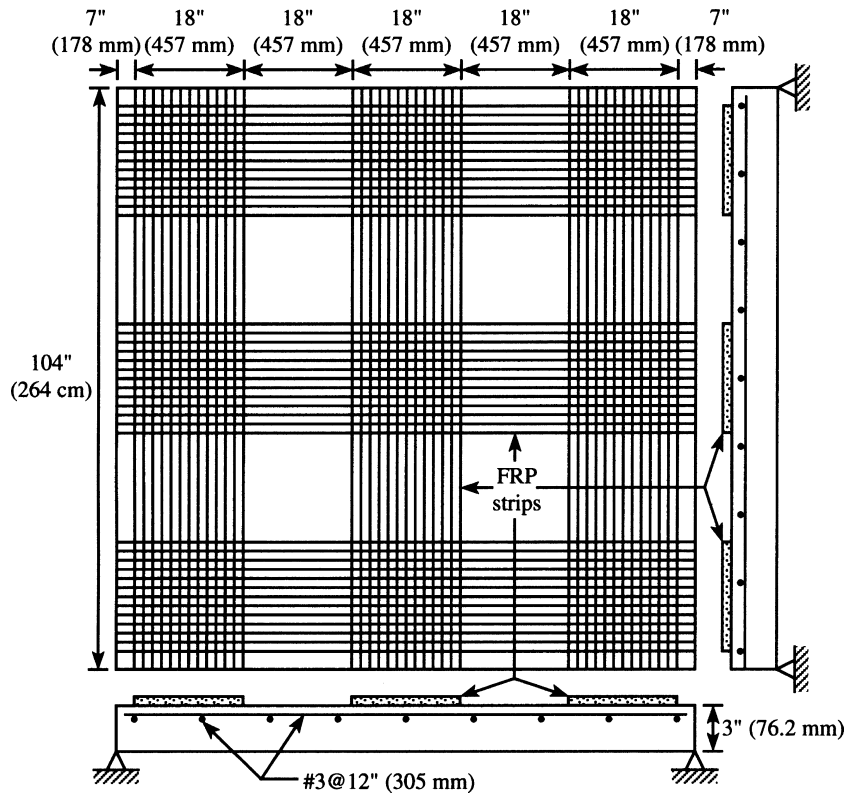


Fig. 15. Mosallam–Mosalam specimens for reinforced concrete two-way slabs.



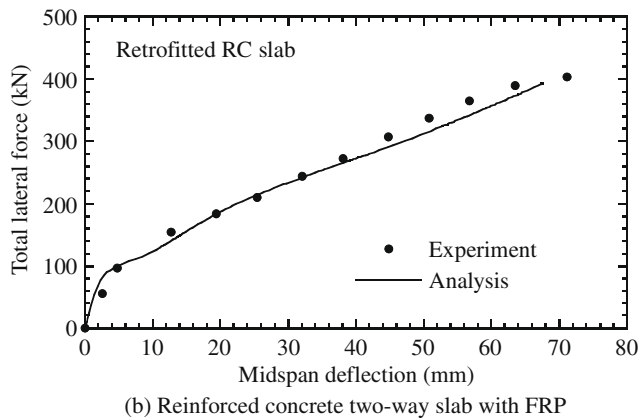
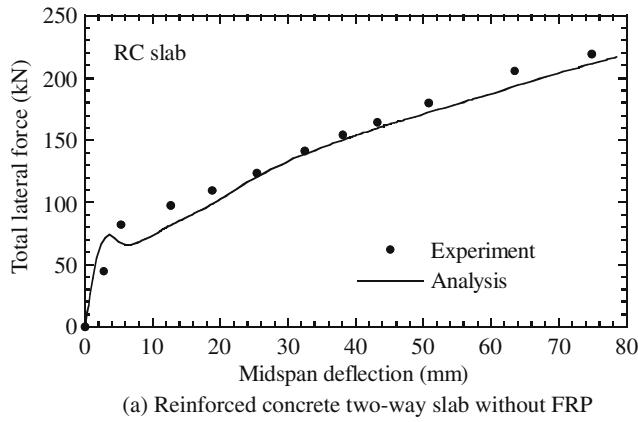


Fig. 16. Load-deflection curves of reinforced concrete two-way slabs with or without FRP.

with the experimental ultimate load 404 kN with an error of 2.5%. This again verifies the validity of the proposed material models to model the behavior of reinforced concrete structures strengthened by FRP.

Comparing Fig. 16b with Fig. 16a, we can find that the ultimate load of the slab is significantly increased by about 84%, which proves the usefulness of FRP to strengthen reinforced concrete structures.

3.6. Laterally loaded prestressed concrete solid slab strengthened by FRP

In this example, prestressed concrete solid slab tested by Shahawy et al. [10] is studied. The test setup for the solid slab is shown in Fig. 17. The compressive strength  $f'_c$  of the concrete is 41.37 MPa. The pretensioned solid slab has 16 low relaxation strands and the diameter of each strand is 13 mm. The ultimate strength of the prestressing steel is 1862 MPa and the jacking force in each strand is maintained at 138 kN. The elastic modulus of the prestressing strands is assumed to be  $E_s = 190$  GPa and the assumed stress-strain relation of the strand is shown in Fig. 18. The FRP layer is 0.17 mm in thickness with tensile strength  $\bar{X} = 2758$  MPa and modulus  $E_{11} = 141.3$  GPa. In addition, the following parameters are assumed:  $\bar{Y} = \bar{S} = 27.5$  MPa,  $\bar{X}' = \bar{Y}' = -27.5$  MPa,  $E_{22} = G_{12} = 1413$  MPa,  $S_{6666} = 0$ ,  $\nu_{12} = 0.21$ .

In the experiment, the prestressed concrete slab is simply supported at two ends (Fig. 17). The slab is subjected to four-point static loading up to its ultimate capacity and unloaded. Then, three FRP layers are bonded to the tension face of the slab. After a complete cure of the adhesive, the retrofitted slab is tested again under four-point loading. Since the retrofitted concrete slab is pre-

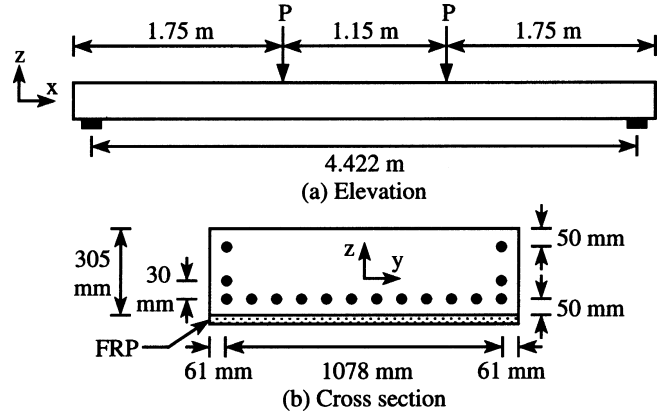


Fig. 17. Shahawy et al. specimen for prestressed concrete solid slab.

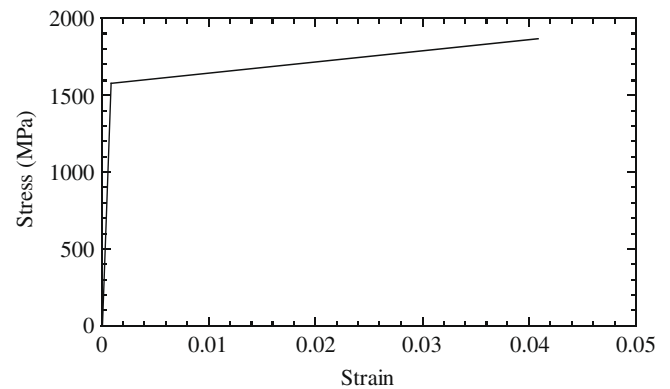


Fig. 18. Stress-strain relation of prestressing steel strand for Shahawy et al. specimen.

cracked, the compressive strength of the precracked concrete should be smaller than that of the intact concrete. Therefore,  $f'_c = 35$  MPa with a 15% reduction [56] is used in the numerical analysis.

The slab has two planes of symmetry. Therefore, only 1/4 portion of the slab is analyzed and symmetric boundary conditions are placed along the two symmetric planes. In the finite element analysis, 8-node solid elements with reduced integration rule are used to model the prestressed concrete slabs. Based on the result of convergence test [57], the finite element mesh has 54 solid elements in total (9 rows in  $x$ -direction, 3 rows in  $y$ -direction, 2 rows in  $z$ -direction). The fiber-reinforced plastics are modeled by the

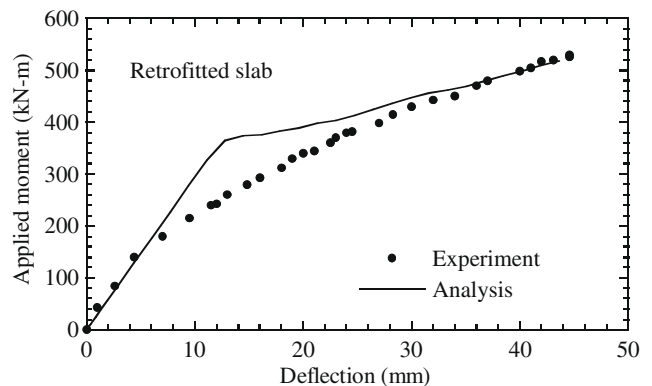


Fig. 19. Moment-deflection curves of prestressed concrete solid slab strengthened by FRP at loading points.

4-node shell elements with reduced integration rule and perfect bonding between FRP and the slab is assumed.

Fig. 19 shows the moment versus deflection curves of the retrofitted prestressed concrete solid slab at the loading points. It can be observed that the correlation is quite good between the numerical result and the experimental data. The predicted ultimate load 519 kN-m is in good agreement with the experimental ultimate load 530 kN-m with an error of 2.1%. This verifies the validity of the proposed material models to model the behavior of prestressed concrete structures strengthened by FRP.

#### 4. Conclusions

As a conclusion, the good agreement obtained in these sets of examples between the numerical predictions and the experimental results establishes the validity and accuracy of using the proposed nonlinear constitutive models for steel reinforcing bars, prestressing tendons, concrete and fiber-reinforced plastics in modeling the behavior of reinforced concrete structures, prestressed concrete structures, reinforced concrete structures strengthened by fiber-reinforced plastics and prestressed concrete structures strengthened by fiber-reinforced plastics.

It should be noted that in these sets of experimental works, the FRP strips are placed either in the longitudinal direction or in the transverse direction on the tension side of the concrete structures. These FRP strips are primarily subjected to uniaxial tensile stresses or biaxial tensile stresses. Consequently, the nonlinear shear effect of FRP is not significant and the numerical results obtained by the nonlinear shear constitutive model are almost the same as those obtained by the linear shear constitutive model. However, the authors' previous study has shown that when the FRP strips are subjected to shear stresses, the nonlinear shear effect is significant. For such cases, nonlinear shear constitutive model for FRP strengthening is an actual improvement [58].

In the analysis, the composite strips are assumed to be perfectly bonded to the concrete surface. This no slip assumption has also been used by many investigators [13,59,60]. Existing test data suggest that the main failure mode is concrete failure, leading to crack propagation parallel to the bonded FRP adjacent to the adhesive-to-concrete interface [61]. A very important aspect of bond behavior is that there exists an effective bond length beyond which an extension of the bond length cannot increase the bond strength as well as the ultimate load of the strengthened concrete structure [61–63]. As long as the criterion of the effective bond length has been fulfilled, it is quite justifiable to use the perfect bonding assumption. The good correlations between the numerical predictions and the experimental results from this investigation demonstrate the validity of the perfect bonding assumption.

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