CONSTITUTIVE MODELING OF CONCRETE BY USING NONASSOCIATED PLASTICITY

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ABSTRACT: In recent years, the theory of plasticity has been widely adopted in the constitutive modeling of plain concrete. Unfortunately, the fundamental hypothesis that forms the basis of associated plasticity does not hold for concrete in some loading conditions, so that the associated plasticity models sometimes lead to great discrepancies between predicted and measured response (Chen 1981; Vermeer and de Borst 1984; Hu and Schnobrich 1988). In order to study a nonassociated plasticity for plane stress-state concrete under short-term monotonic loading, an elastic strain-hardening plastic model is derived herein. This concrete material model has been tested against the experimental results of Kupfer et al. (1969). It has been demonstrated that the results achieved using an associated flow rule are usually poor, while the predictions based on the nonassociated flow rule show very good agreement with the test data.

INTRODUCTION

When concrete is subjected to compressive stresses, experimental results (Sinha et al. 1964) have indicated that the nonlinear deformations of concrete are basically inelastic because upon unloading only a portion of those strains can be recovered from the total strains (Fig. 1). Therefore, the stress-strain behavior of the concrete material may be separated into recoverable and nonrecoverable components. The recoverable part can be treated within the field of elasticity theory, while the irrecoverable part can be treated by the theory of plasticity. Plasticity-based models have been used extensively in recent years to describe the behavior of concrete. In general, models based on the theory of plasticity describe concrete as an elastic-perfectly plastic material (Mikkola and Schnobrich 1970; Hand et al. 1972; Abdel Rahman 1982), or, to account for the hardening behavior up to the ultimate strength, as an elastic strain-hardening plastic material (Chen and Chen 1975; Buyukozturk 1977; Murray et al. 1979; Chen and Ting 1980; Schnobrich and Hu 1985; Hen and Chen 1987; Hu and Schnobrich 1988).

Since the elastic strain-hardening plastic model is more general and more accurate than the earlier elastic-perfectly plastic models, it is used in this investigation. In order to apply the incremental theory of elastic strain-hardening plasticity, several aspects must be specified beforehand. These include: (1) The yield function that defines the initial and subsequent yield surfaces; (2) the hardening rules that describe the motion of the subsequent yield surface during continuous loading; (3) the flow rules that relate the plastic strain increments to stress increments; (4) the equivalent uniaxial stress-strain curve; and (5) the plastic hardening modulus. All of these will be briefly discussed in the following.

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Yield Functions

For strain-hardening concrete, the subsequent yield surfaces change during continued straining beyond the initial yield surface. In plasticity theory, it is convenient to assume that the initial and subsequent yield conditions can be defined by the same yield function expressed in the following form:

$$f(\{\sigma\},\underline{\sigma}) = F(\{\sigma\}) - \underline{\sigma} = 0 \qquad (1)$$

such that whenever the function F becomes equal to the value of $\underline{\sigma}$ yielding would occur, at which time $\underline{\sigma}$ then takes on a new value. $\{\sigma\} = \{\sigma_x, \sigma_y, \tau_{xy}\}$ is a stress vector. The function F can be looked upon as a loading function; and $\underline{\sigma}$ = a hardening parameter called the "equivalent stress." The parameter $\underline{\sigma}$ depends on the complete previous stress and strain history of the material and its strain-hardening properties. In this paper, the yield functions proposed by Hand et al. (1972) are used with slight modifications (Hu and Schnobrich 1988). These yield functions are defined in the context of biaxial tension, combined tension-compression, and biaxial compression, separately.

For biaxial tension, it is assumed that the initial yield surface coincides with the failure surface (Fig. 2). Under this assumption, concrete behaves in a purely linear elastic fashion up to failure with no plastic deformation having occurred. The failure surface for biaxial tension in this investigation is defined as

$$f = c_1\left(\frac{3}{2\sqrt{2}}\frac{1+\alpha}{\alpha}\tau_{oct} + \frac{3}{2}\frac{1-\alpha}{\alpha}\sigma_m\right) - f_c' = 0 \qquad (2)$$

where f_c' = the maximum compressive strength of concrete; and σ_m and τ_{oct} = the mean stress and the octahedral shear stress, respectively. For plane stress conditions, they have the following forms:

$$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y) \qquad (3)$$

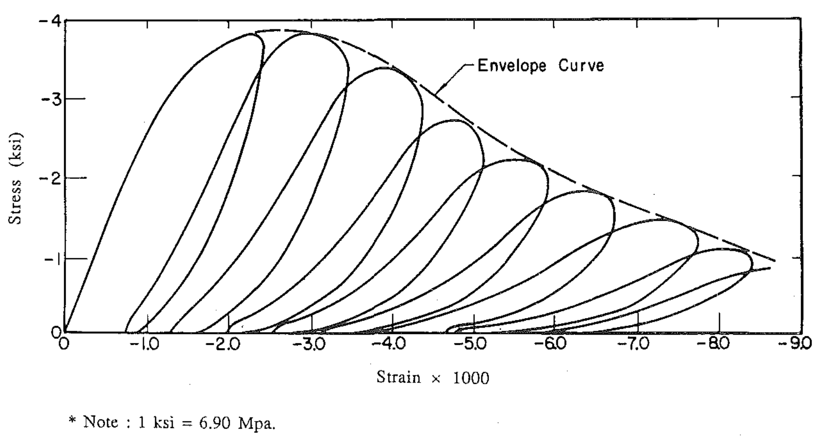

FIG. 1. Behavior of Concrete under Cycles of Compressive Loading (Sinha et al. 1964)

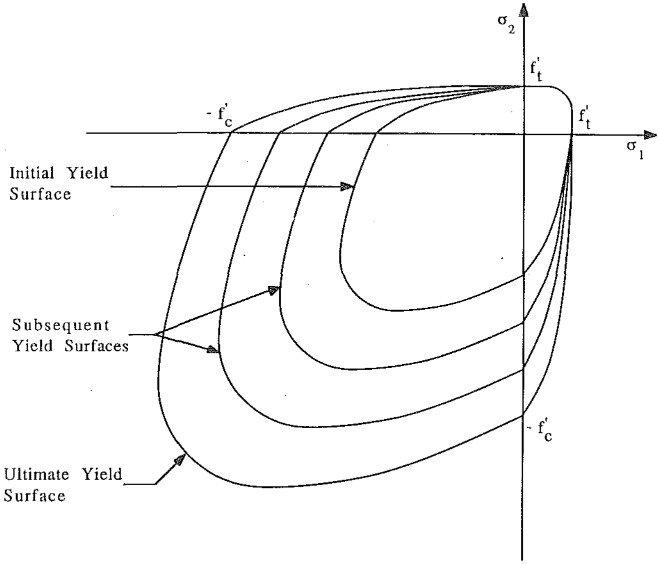

FIG. 2. Yield Surface of Concrete in Two-Dimensional Principal Stress Plane

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2} \quad (4)$$

In Eq. 2, $\alpha = f_t'/f_c' \approx 0.09$, in accordance with the experimental results (Kupfer et al. 1969), where f_t' = the maximum tensile strength of concrete. The variable c_1 can be determined by the following equation

$$c_1 = 1 - 0.4019\left(\frac{\sigma_2}{\sigma_1}\right) + 0.008913\left(\frac{\sigma_2}{\sigma_1}\right)^2 \quad (5)$$

where σ_1 and σ_2 = principal stresses, with $\sigma_1 \geq \sigma_2$.

When concrete is subjected to a combined tension-compression stress state, the yield function is defined as

$$f = c_2\left(\frac{3}{2\sqrt{2}}\frac{1+\alpha}{\alpha}\tau_{oct} + \frac{3}{2}\frac{1-\alpha}{\alpha}\sigma_m\right) - \underline{\sigma} = 0 \quad (6)$$

where for $-\infty < \sigma_1/\sigma_2 < -0.103$ (with σ_1 having positive value and σ_2 having negative value)

$$c_2 = 1 - 0.02886\left(\frac{\sigma_2}{\sigma_1}\right) - 0.006657\left(\frac{\sigma_2}{\sigma_1}\right)^2 - 0.0002443\left(\frac{\sigma_2}{\sigma_1}\right)^3 \quad (7)$$

and for $-0.103 \leq \sigma_1/\sigma_2 < 0$

$$c_2 = 1 + 6.339\left(\frac{\sigma_1}{\sigma_2}\right) + 68.82\left(\frac{\sigma_1}{\sigma_2}\right)^2 + 183.8\left(\frac{\sigma_1}{\sigma_2}\right)^3 \quad (8)$$

For biaxial compression, the yield function is defined as

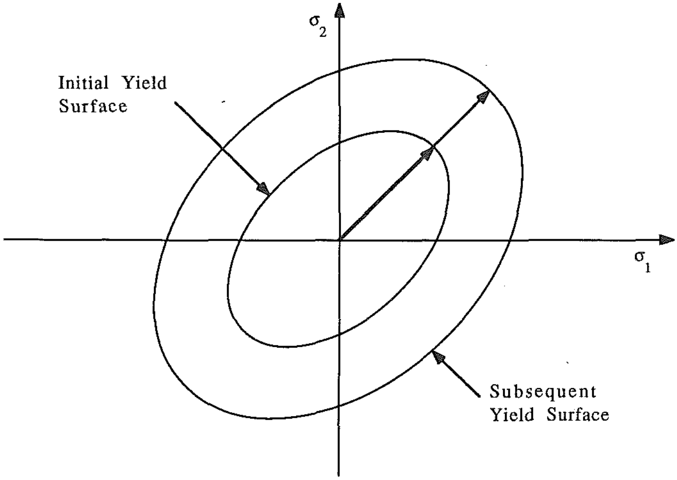

FIG. 3. Isotropic Hardening Rule

$$f = c_3\left(\frac{3}{\sqrt{2}}\,\frac{2\beta - 1}{\beta}\,\tau_{oct} + 3\,\frac{\beta - 1}{\beta}\,\sigma_m\right) - \underline{\sigma} = 0 \quad (9)$$

where $\beta = 1.16$ and

$$c_3 = 1 + 0.05848\left(\frac{\sigma_2}{\sigma_1}\right) - 0.05848\left(\frac{\sigma_2}{\sigma_1}\right)^2 \quad (10)$$

with σ_1 and σ_2 both having negative values, and $\sigma_1 \geq \sigma_2$.

Hardening Rule

The hardening rule defines the motion of the subsequent yield surfaces during plastic loading. A number of hardening rules have been proposed, such as isotropic hardening, kinematic hardening, and mixed hardening rules (Chen 1982; Mendelson 1983). Among these three hardening rules, the assumption of isotropic hardening is the simplest one to formulate mathematically; it is used in this study. This hardening rule assumes that the yield surface expands uniformly without distortion as plastic deformation occurs, as shown schematically in Fig. 3. It is known that the Bauschinger effect cannot be modeled by the isotropic hardening rule. However, under the monotonic loading condition, the Bauschinger effect is not crucial because no reverse loading takes place. As a consequence, the isotropic hardening rule is adequate in modeling the hardening behavior of concrete under the monotonic loading condition.

Flow Rules and Drucker's Instability Postulates

The total strains $\{\epsilon\} = \{\epsilon_x, \epsilon_y, \gamma_{xy}\}$ experienced by a plastic body can be divided into the sum of the elastic and plastic strains, $\{\epsilon\}_e$ and $\{\epsilon\}_p$, i.e.

$$\{\epsilon\} = \{\epsilon\}_e + \{\epsilon\}_p \qquad (11)$$

If considered in terms of incremental strains, then

$$d\{\epsilon\} = d\{\epsilon\}_e + d\{\epsilon\}_p \qquad (12)$$

When the concrete deforms plastically, it is conventional to assume that, based on the normality condition, the incremental plastic strains $d\{\epsilon\}_p$ can be related to a plastic potential function

$$g(\{\sigma\},\underline{\sigma}) = G(\{\sigma\}) - \underline{\sigma} = 0 \qquad (13)$$

by the following equation:

$$d\{\epsilon\}_p = \lambda \frac{\partial g}{\partial\{\sigma\}} = \lambda \frac{\partial G}{\partial\{\sigma\}} \qquad (14)$$

where λ = a positive scalar factor that may vary through the hardening process. The gradient of the potential surface $\partial g/\partial\{\sigma\}$ defines the direction of the incremental plastic strain vector $d\{\epsilon\}_p$, and the length is determined by the factor λ. Because the vector $\partial g/\partial\{\sigma\}$ is normal to the potential surface, it is easy to see that the incremental plastic strain is also normal to the surface defined by the plastic potential function g. This condition is called the normality law. In the simplest case when the plastic potential function and yield function coincide ($g = f$), then

$$d\{\epsilon\}_p = \lambda \frac{\partial f}{\partial\{\sigma\}} = \lambda \frac{\partial F}{\partial\{\sigma\}} \qquad (15)$$

Eq. 15 is called the associated flow rule because the incremental plastic strains are connected with the yield function f. If $g \neq f$, Eq. 14 is termed a nonassociated flow rule.

Use of the associated flow rule satisfies Drucker's local material instability postulates (Drucker 1950, 1951). It has been implemented in concrete material models by many investigators (Mikkola and Schnobrich 1970; Hand et al. 1972; Chen and Chen 1975; Buyukozturk 1977; Abdel Rahman 1982; Murray et al. 1979). However, it has been found that the associated flow rule does not hold for the whole range of the response spectrum of concrete and that it sometimes leads to great discrepancies between predicted and measured response, as well as load-carrying capacities (Chen 1981; Vermeer and de Borst 1984; Hu and Schnobrich 1988).

On the other hand, using a nonassociated flow rule might violate Drucker's local material instability postulates. However, whereas these postulates provide sufficient conditions for stability, it has been suggested that they are not necessary conditions (Mroz 1963). In the past, a number of constitutive equations have been proposed that do not obey the associated flow rule, and satisfactory results have been obtained (Bodner and Partom 1975). Recently, experimental results have shown that while a granular material dilated during the triaxial compression test, it followed a nonassociated flow rule (Lade et al. 1987). Stable behavior has been observed in a stress region in which Drucker's local material instability postulates were violated and that stability was maintained until the failure surface was reached. Similarly to granular material, such a dilatant volume increase is also observed for concrete (Kupfer, et al., 1969). By an evaluation of existing test data, the need for nonassociated plasticity for concrete has been demonstrated (Vermeer and de Borst 1984).

In this investigation, both associated and nonassociated flow rules have been used to formulate the constitutive equations for concrete. For a non-associated flow rule, the simplest von Mises yield function is used. It has the following form:

$$g = \frac{3}{\sqrt{2}} \tau_{oct} - \underline{\sigma} = 0 \quad (16)$$

EQUIVALENT UNIAXIAL STRESS-STRAIN CURVE

When plastic deformation occurs, there should be a certain parameter to guide the expansion of the yield surface. A commonly used approach is to relate the multidimensional stress and strain conditions to a pair of quantities, namely, the equivalent stress $\underline{\sigma}$ and equivalent strain $\underline{\epsilon}$, such that results obtained following different loading paths can all be correlated by means of the equivalent uniaxial stress-strain curve.

The uniaxial compressive stress-strain relationship proposed by Saenz (1964) is now widely accepted as the mathematical description of the uniaxial stress-strain curve for concrete. However, it has been found that for different principal stress ratios, the corresponding equivalent uniaxial stress-strain curves were quite different and had a large variety (Chen and Chen 1975; Buyukozturk 1977). In this study, in order to make the equivalent uniaxial stress-strain curve more general, a variable q, dependent on the principal stress ratio, is implicitly added into Saenz's equation (Hu and Schnobrich 1988). The equivalent uniaxial stress-strain curve then has the following form:

$$\underline{\sigma} = \frac{E_c \underline{\epsilon}}{1 + (R + R_E - 2)\left(\frac{\underline{\epsilon}}{\epsilon_*}\right) - (2R - 1)\left(\frac{\underline{\epsilon}}{\epsilon_*}\right)^2 + R\left(\frac{\underline{\epsilon}}{\epsilon_*}\right)^3} \quad (17)$$

where

$$R = \frac{R_E(R_\sigma - 1)}{(R_\epsilon - 1)^2} - \frac{1}{R_\epsilon} \quad (18)$$

is the ratio relation; $R_E = E_c/E_o$ = the modular ratio; $R_\sigma = f'_c/\sigma_f$ = the stress ratio; $R_\epsilon = \epsilon_f/\epsilon_*$ = the strain ratio; and $E_o = f'_c/\epsilon_*$ = the secant modulus. In Eq. 17, $\epsilon_* = q\epsilon_o$ = the strain corresponding to f'_c on the equivalent uniaxial stress-strain curve; ϵ_o = the strain corresponding to f'_c in an uniaxial compression test; E_c = the initial modulus of elasticity; and ϵ_f and σ_f = the maximum strain and the corresponding stress on the equivalent uniaxial stress-strain curve (Fig. 4).

The value of the variable q can be determined as follows (with $\sigma_1 \geq \sigma_2$). In the combined tension-compression region, for $-\infty < \sigma_1/\sigma_2 < -0.103$

$$q = \frac{f'_c}{E_c\epsilon_o} + \left(1 - \frac{f'_c}{E_c\epsilon_o}\right)\left[0.001231\left(\frac{\sigma_2}{\sigma_1}\right) + 0.001469\left(\frac{\sigma_2}{\sigma_1}\right)^2 + 0.00001340\left(\frac{\sigma_2}{\sigma_1}\right)^3\right] \quad (19)$$

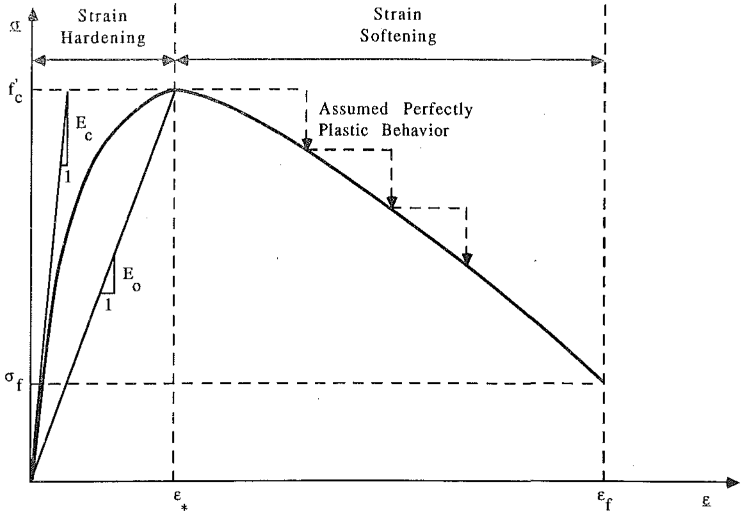

FIG. 4. Equivalent Uniaxial Stress-Strain Curve for Concrete

for $-0.103 \leq \sigma_1/\sigma_2 < 0$

$$q = \frac{f'_c}{E_c\epsilon_o} + \left(1 - \frac{f'_c}{E_c\epsilon_o}\right)\left[1 + 13.96\left(\frac{\sigma_1}{\sigma_2}\right) + 59.21\left(\frac{\sigma_1}{\sigma_2}\right)^2 + 69.24\left(\frac{\sigma_1}{\sigma_2}\right)^3\right] \quad (20)$$

In the biaxial compression region

$$q = \frac{f'_c}{E_c\epsilon_o} + \left(1 + \frac{f'_c}{E_c\epsilon_o}\right)\left[1 + 1.782\left(\frac{\sigma_1}{\sigma_2}\right) + 0.5936\left(\frac{\sigma_1}{\sigma_2}\right)^2\right] \quad (21)$$

For the values of σ_f and ϵ_f, Darwin and Pecknold (1974) used $R_\sigma = 5$, $R_\epsilon = 4$; Elwi and Murray (1979) and Chen (1981) used $R_\sigma = 4$, $R_\epsilon = 4$. Generally, to define σ_f and ϵ_f on any rigorous experimental basis is impossible, because the descending branch of the stress-strain curve is highly test-dependent and is usually unavailable from statically determinate tests. In this study, it is assumed that $R_\sigma = 4$ and $R_\epsilon = 4$.

The equivalent uniaxial tangent modulus E_t can be calculated by differentiating Eq. 17 with respect to the equivalent strain $\underline{\epsilon}$. It has the following form:

$$E_t = \frac{d\underline{\sigma}}{d\underline{\epsilon}} = \frac{E_c\left[1 + (2R - 1)\left(\frac{\underline{\epsilon}}{\epsilon_*}\right)^2 - 2R\left(\frac{\underline{\epsilon}}{\epsilon_*}\right)^3\right]}{\left[1 + (R + R_E - 2)\left(\frac{\underline{\epsilon}}{\epsilon_*}\right) - (2R - 1)\left(\frac{\underline{\epsilon}}{\epsilon_*}\right)^2 + R\left(\frac{\underline{\epsilon}}{\epsilon_*}\right)^3\right]^2} \quad (22)$$

Beyond the peak stress point in the strain-softening region, with further straining, the compressive stress begins to decrease and the equivalent uniaxial tangent modulus becomes negative. In order to prevent the numerical difficulties associated with a negative tangent modulus, once the ultimate yield stress f'_c has been reached, E_t is set to zero and concrete then behaves like a perfectly plastic material. This plastic response is allowed to propagate through a limited strain $\Delta\epsilon$, at which time the unbalanced stress is released. This process follows in a stepwise fashion, with the corresponding yield surface being contracted simultaneously.

Plastic-Hardening Modulus

After the equivalent uniaxial stress-strain curve is built up, the next step is to find the plastic-hardening modulus H, which is used to control the movement of the yield surface. This modulus is defined as

$$H = \frac{d\underline{\sigma}}{d\underline{\epsilon}_p} \qquad (23)$$

where $d\underline{\sigma}$ = the incremental equivalent stress; and $d\underline{\epsilon}_p$ = the incremental equivalent plastic strain. Before H is calculated, it is important to notice that the equivalent strain $\underline{\epsilon}$ is made of an elastic part $\underline{\epsilon}_e$ and a plastic part $\underline{\epsilon}_p$

$$\underline{\epsilon} = \underline{\epsilon}_e + \underline{\epsilon}_p \qquad (24)$$

If in terms of incremental strains, then

$$d\underline{\epsilon} = d\underline{\epsilon}_e + d\underline{\epsilon}_p \qquad (25)$$

By dividing Eq. 25 by $d\underline{\sigma}$ and using Eqs. 22 and 23, this incremental strain equation becomes

$$\frac{1}{E_t} = \frac{1}{E_c} + \frac{1}{H} \qquad (26)$$

Rearranging Eq. 26, the plastic-hardening modulus can be expressed as

$$H = \frac{E_c E_t}{(E_c - E_t)} \qquad (27)$$

The total equivalent plastic strain can now be calculated by integrating the equivalent incremental plastic strain as follows:

$$\underline{\epsilon}_p = \int d\underline{\epsilon}_p = \int \frac{d\underline{\sigma}}{H} \qquad (28)$$

Constitutive Equations for Concrete

Once the hardening rule has been selected and the yield function has been defined, incremental plastic stress-strain relations based on the flow rules as described are applicable to such a material model. Through the plastic-hardening modulus, the corresponding constitutive equations can then be derived. Now consider Eq. 1:

$$f(\{\sigma\}, \underline{\sigma}) = 0$$

Differentiating Eq. 1, we have

$$df = \frac{\partial f}{\partial\{\sigma\}} d\langle\sigma\rangle + \frac{\partial f}{\partial\underline{\sigma}} \frac{d\underline{\sigma}}{d\underline{\epsilon}_p} d\underline{\epsilon}_p = 0 \qquad (29)$$

From the theory of elasticity, the incremental stresses can be related to the incremental elastic strains by a generalized Hooke's law. In abbreviated form

$$d\langle\sigma\rangle = [C]_e d\langle\epsilon\rangle_e = [C]_e(d\langle\epsilon\rangle - d\langle\epsilon\rangle_p) \quad (30)$$

in which $[C]_e$ = a 3 × 3 elastic material property matrix. Within the context of plane stress, this matrix is given as follows:

$$[C]_e = \frac{E_c}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \quad (31)$$

where ν = the Poisson's ratio of concrete. Substituting Eqs. 23 and 30 into Eq. 29, then

$$\frac{\partial f}{\partial\{\sigma\}}[C]_e(d\langle\epsilon\rangle - d\langle\epsilon\rangle_p) + \frac{\partial f}{\partial \underline{\sigma}} H d\underline{\epsilon}_p = 0 \quad (32)$$

From Eqs. 14 and 15 and given that $\partial f/\partial\underline{\sigma} = -1$, Eq. 32 becomes

$$\frac{\partial F}{\partial\{\sigma\}}[C]_e\left(d\langle\epsilon\rangle - \lambda\frac{\partial G}{\partial\langle\sigma\rangle}\right) - H d\underline{\epsilon}_p = 0 \quad (33)$$

Consider the plastic work done during the plastic deformation

$$dW_p = \underline{\sigma} d\underline{\epsilon}_p = \{\sigma\} d\langle\epsilon\rangle_p \quad (34)$$

Following a rearrangement of Eq. 34, the increment in plastic strain is

$$d\underline{\epsilon}_p = \frac{\{\sigma\}d\langle\epsilon\rangle_p}{\underline{\sigma}} = \frac{\{\sigma\}\lambda\dfrac{\partial G}{\partial\langle\sigma\rangle}}{\underline{\sigma}} = \lambda\frac{G}{\underline{\sigma}} \quad (35)$$

Substituting Eq. 35 into Eq. 33 and solving for λ, we obtain

$$\lambda = \frac{\dfrac{\partial F}{\partial\{\sigma\}}[C]_e d\langle\epsilon\rangle}{H\dfrac{G}{\underline{\sigma}} + \dfrac{\partial F}{\partial\{\sigma\}}[C]_e\dfrac{\partial G}{\partial\langle\sigma\rangle}} \quad (36)$$

Finally, if Eq. 36 is substituted into Eq. 30, then the incremental stress-strain constitutive equation for concrete can be expressed as

$$d\langle\sigma\rangle = [C]_{ep} d\langle\epsilon\rangle = ([C]_e - [C]_p) d\langle\epsilon\rangle \quad (37)$$

where

$$[C]_p = \frac{[C]_e\dfrac{\partial G}{\partial\langle\sigma\rangle}\dfrac{\partial F}{\partial\{\sigma\}}[C]_e}{H\dfrac{G}{\underline{\sigma}} + \dfrac{\partial F}{\partial\{\sigma\}}[C]_e\dfrac{\partial G}{\partial\langle\sigma\rangle}} \quad (38)$$

There are several things worth noting in Eqs. 37 and 38. First, if an associated flow rule is used (i.e., $f = g$ and $F = G$), then $[C]_{ep}$ is symmetrical. Otherwise, $[C]_{ep}$ becomes unsymmetrical, and, in order to carry out a finite element solution, an unsymmetrical equation solver is needed. Second, under the condition that the plastic hardening modulus $H = 0$, the yield surface does not expand regardless of how much the load is increased. Eq. 37 then becomes the incremental stress-strain relationship for an elastic-perfectly plastic material and $[C]_{ep}$ is singular.

Comparison with Experimental Results

In order to test the proposed elastic strain-hardening plastic model for plain concrete, the experimental data of Kupfer et al. (1969) has been selected as the basis for a comparison. In the numerical analysis, a nine-node Lagrangian shell element is employed, and a full Gaussian integration (3 × 3) rule is used for that element to avoid the possibility of any zero energy modes. The loading directions and the finite element idealization for the specimens are shown in Fig. 5. Details of the material properties used in the numerical examples are given in Table 1.

The predicted responses of concrete in the biaxial tension stress region are

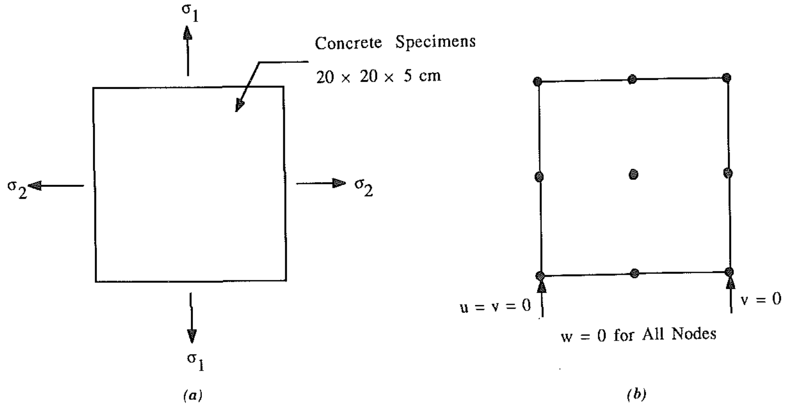

FIG. 5. Kupfer-Hilsdorf-Rusch Specimens: (*a*) Plan View of Specimens and Loading Directions; (*b*) Finite Element Idealization

TABLE 1. Material Properties for Kupfer-Hilsdorf-Rusch Specimens

$\sigma_1:\sigma_2$ (1)	f'_c (psi) (2)	f'_t (psi) (3)	E_c (ksi) (4)	ϵ_o (5)	ν (6)
−1:−1	4,650	419	4,200	0.0022	0.20
−1:−0.52	4,650	419	4,200	0.0022	0.20
−1:0	4,650	419	4,200	0.0022	0.20
−1:0.052	4,650	419	4,200	0.0022	0.19
−1:0.103	4,650	419	4,200	0.0022	0.19
−1:0.204	4,650	419	4,200	0.0022	0.19
1:0	4,200	378	4,550	—	0.18
1:0.55	4,200	378	4,550	—	0.18
1:1	4,200	378	4,550	—	0.18

Note: 1 psi = 0.00690 MPa; 1 ksi = 6.90 MPa.

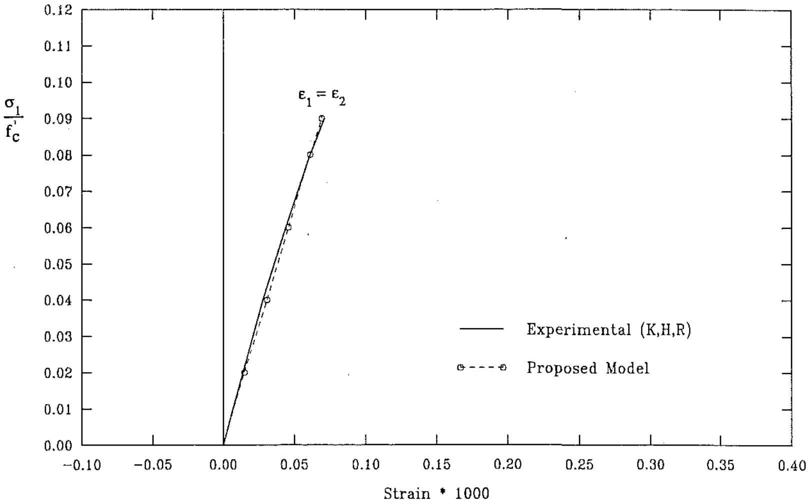

FIG. 6. Comparison of Proposed Model with Biaxial Tension Test $\sigma_1/\sigma_2 = 1/1$

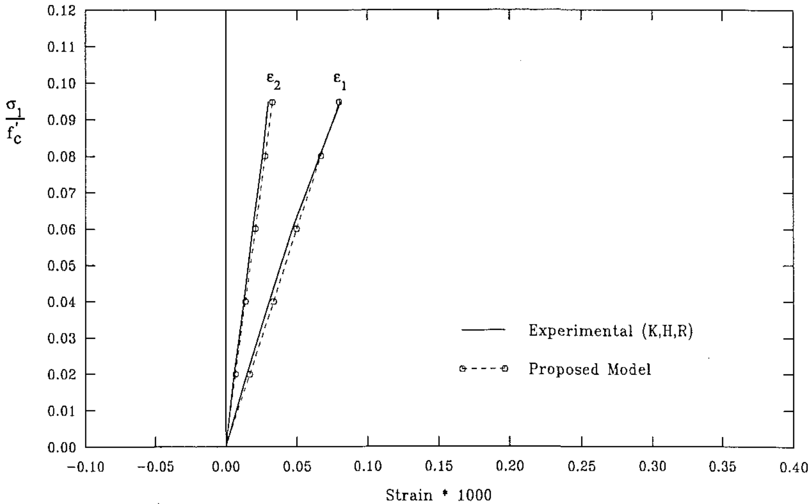

FIG. 7. Comparison of Proposed Model with Biaxial Tenstion Test $\sigma_1/\sigma_2 = 1/0.55$

plotted in Figs. 6–8. Because there are no plastic deformations occurring during the loading process, concrete is observed to behave in a purely linear elastic mode up to failure. The proposed linear elastic model in this stress region works very well, and excellent agreement is obtained. In this paper, only the behavior of concrete prior to failure is studied, and the load-controlled method is used to obtain the numerical solutions. If the displacement method is employed, then the tension-stiffening phenomenon, the shear-retention factor, and the stress-degrading effect for concrete parallel to the

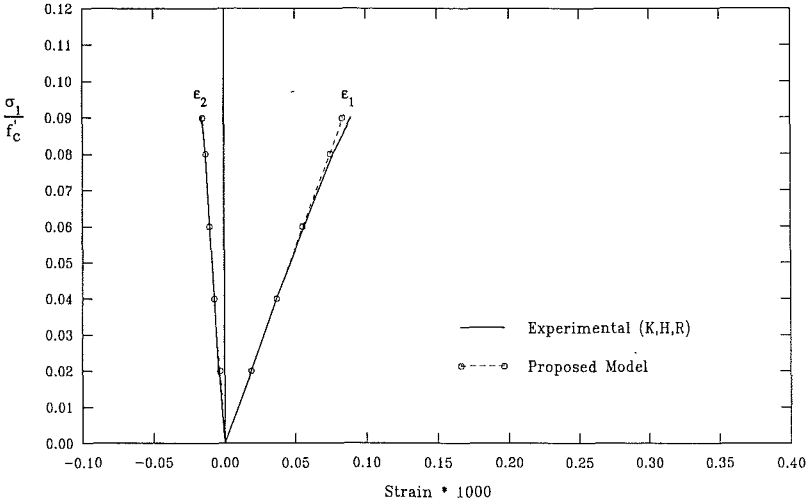

FIG. 8. Comparison of Proposed Model with Biaxial Tension Test σ_1/σ_2 = 1/0

crack direction must be included in the material model (Hu and Schnobrich 1988), so that the post-cracking behavior of concrete can be studied.

For concrete subjected to combined tension-compression, the computed responses are plotted in Figs. 9–11. It can be seen that the results predicted by the associated flow rule are only good in the major principal direction. In the minor principal direction, the predicted responses are too soft, often moving far from the test data. This phenomenon has also been reported by

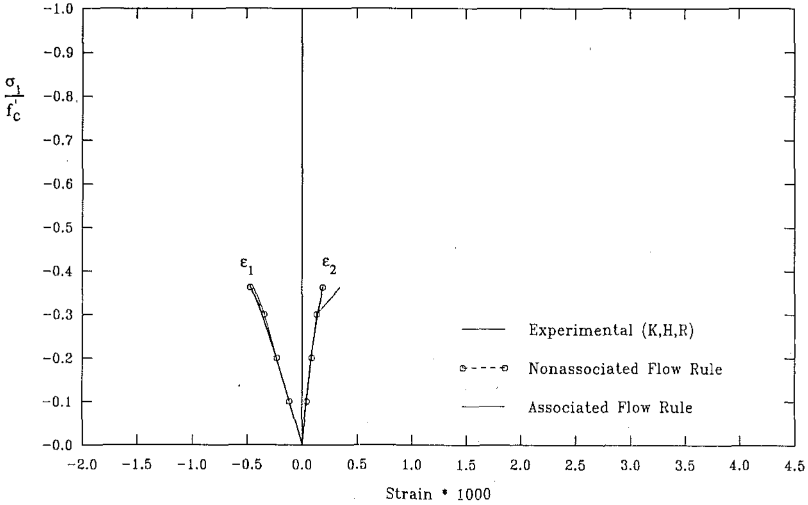

FIG. 9. Comparison of Proposed Model with Combined Tension-Compression Test σ_1/σ_2 = −1/0.204

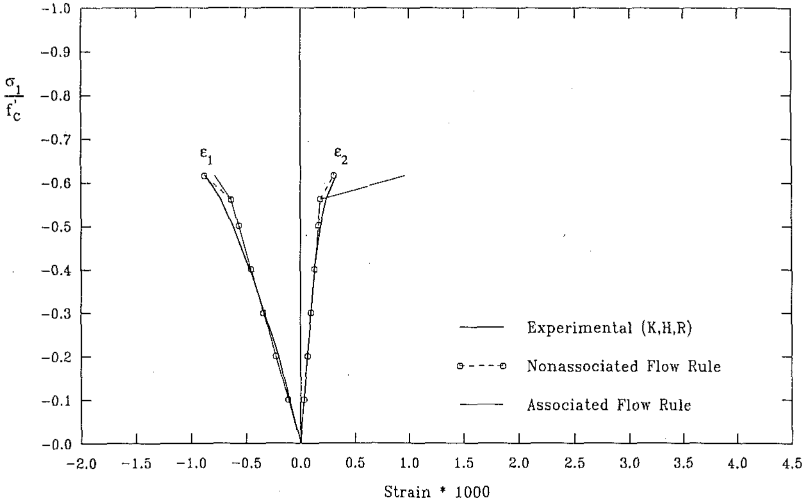

FIG. 10. Comparison of Proposed Model with Combined Tension-Compression Test $\sigma_1/\sigma_2 = -1/0.103$

Murray et al. (1979). On the other hand, the predictions based on a nonassociated flow rule, in which the von Mises yield function is used as the plastic potential function, are in good agreement with the experimental data, not only in the major principal direction but also in the minor principal direction. It appears that the associated flow rule is too restrictive for concrete, especially when the stress combinations involve the high compression and low tension region.

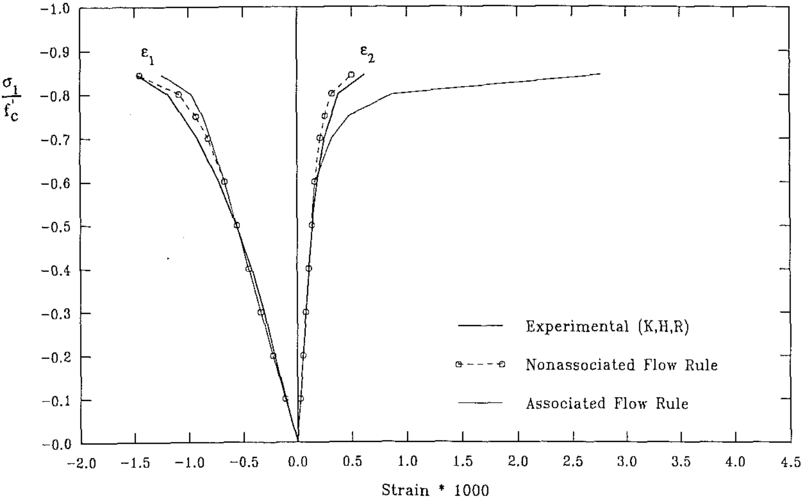

FIG. 11. Comparison of Proposed Model with Combined Tension-Compression Test $\sigma_1/\sigma_2 = -1/0.052$

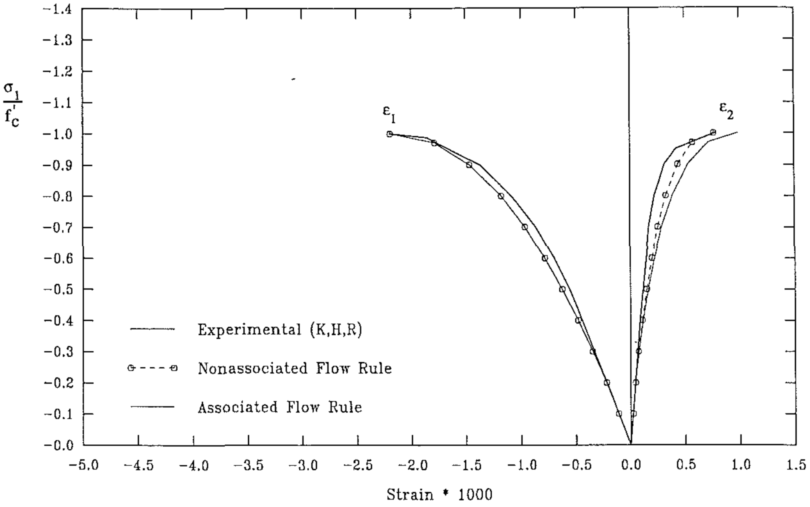

FIG. 12. Comparison of Proposed Model with Biaxial Compression Test $\sigma_1/\sigma_2 = -1/0$

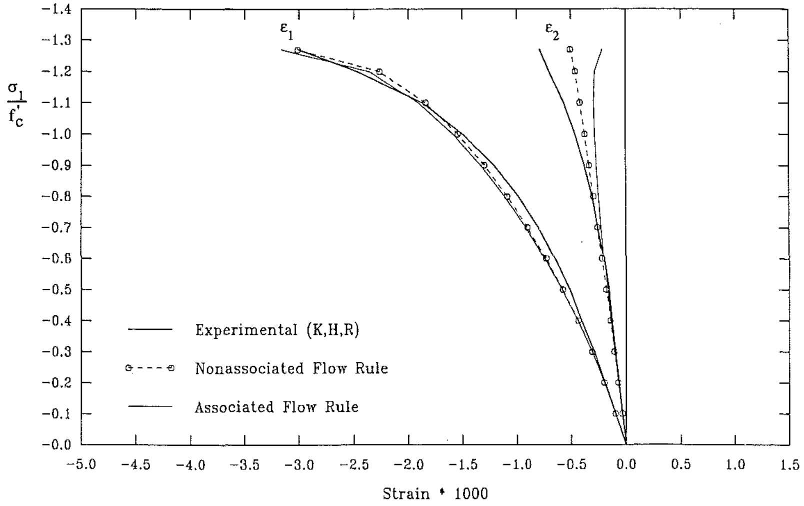

FIG. 13. Comparison of Proposed Model with Biaxial Compression Test $\sigma_1/\sigma_2 = -1/-0.52$

When concrete stresses are in the biaxial compression region, the numerical simulations produce the results plotted in Figs. 12–14. Similarly to the results obtained in the combined tension-compression cases, the predictions computed by the nonassociated flow rule are better than those calculated by an associated flow rule. However, because the directions of the flow vectors formulated by both flow rules are very close, the discrepancy between using

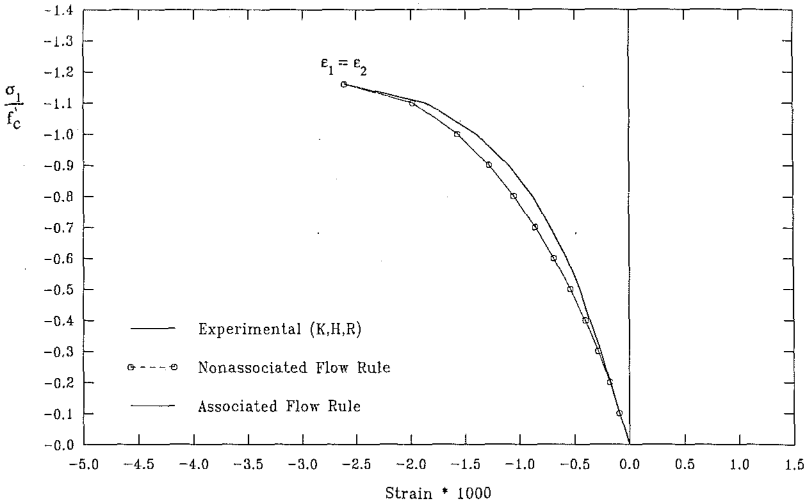

FIG. 14. Comparison of Proposed Model with Biaxial Compression Test $\sigma_1/\sigma_2 = -1/-1$

these two rules is not very large. For the special case with equal biaxial compression, the predictions are the same using either rule, because the two flow vectors lie in the same direction.

As a result of the numerical analysis, it has been found that the proposed elastic strain-hardening plastic model, which includes proposed yield functions, an isotropic hardening formulation, a nonassociated flow rule, and a generalized equivalent uniaxial stress-strain curve, is adequate in describing the plastic behavior of plain concrete.

Symmetrization of Stiffness Matrix

Because of the use of a nonassociated flow rule, the structure stiffness matrix is unsymmetric. In order to save data storage memory and the computing time in decomposing the stiffness matrix, a symmetrization of the stiffness matrix was attempted.

When the principal of virtual work is used to derive a finite element solution, the total virtual work δW (the sum of the external virtual work and the internal virtual work) of an equilibrium system is equal to zero. We can write

$$\delta W = \sum_i \underline{u}_i f_i - \int_v \{\underline{\epsilon}\}\langle\sigma\rangle dv = \sum_i \underline{u}_i f_i - \int_v \{\underline{\epsilon}\}[C]\langle\epsilon\rangle dv = 0 \quad (39)$$

in which $\underline{u}_i$ and $\{\underline{\epsilon}\}$ = nodal virtual displacements and corresponding virtual strains that satisfy the compatibility condition; f_i and $\langle\sigma\rangle$ = the generalized nodal point forces and corresponding stresses satisfying the equilibrium condition; $\langle\epsilon\rangle$ = the strains produced by the stresses $\{\sigma\}$; and $[C]$ = an unsymmetric 3 × 3 material constitutive matrix.

Another way to formulate finite element equations is to use the principal of minimum potential energy. The total potential energy Π of a body is the sum of the strain energy plus the load potential. It can be written as

$$\Pi = \frac{1}{2}\int_v \{\epsilon\}[C]\langle\epsilon\rangle dv - \sum_i u_i f_i \quad (40)$$

where u_i = the nodal displacements. Applying the variational operator δ to Eq. 40 and invoking the stationary condition $\delta\Pi = 0$, we obtain

$$\sum_i \delta u_i f_i - \int_v \left[\delta\{\epsilon\}\frac{1}{2}([C] + [C]^T)\langle\epsilon\rangle\right] dv = 0 \quad (41)$$

Compare Eq. 39 with Eq. 41 and let $\{\underline{\epsilon}\} = \delta\{\epsilon\}$, $\underline{u}_i = \delta u_i$. Then a symmetrized material constitutive matrix $[C]_s$ can be achieved by using

$$[C]_s = \frac{1}{2}[[C] + [C]^T] \quad (42)$$

Eq. 42 has been used to analyze the test data of Kupfer et al. (1969) again. However, the numerical predictions computed by the symmetrized coefficient matrices are poorer than those obtained while using the original unsymmetric coefficient matrices. Furthermore, the convergence is very slow, with the solutions sometimes even diverging. This phenomenon has also been observed by Li et al. (1986) during the process of symmetrization of the stiffness matrix. As a conclusion, the results obtained on the basis of the symmetrized matrix are poor and must be considered unacceptable.

Summary and Conclusions

Based on the theory of nonassociated plasticity, an elastic strain-hardening plastic model for plane stress-state concrete under short-term monotonic loading has been developed, and, as a result, a set of constitutive equations suitable for the incremental finite element analysis has been derived. Features of the present model include various yield functions, an isotropic hardening rule, a nonassociated flow rule, and a generalized equivalent stress-strain curve.

This material model has been tested against the experimental data of Kupfer et al. (1969). It has been demonstrated that this material model is adequate in describing the plastic behavior of plain concrete. Further, it has been shown that the predictions based on the nonassociated flow rule show very good agreement with the test data, while the results achieved using an associated flow rule are poor, especially in regions of combined high compression with low tension.

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Appendix I. References

Abdel Rahman, H. H. (1982). "Computational models for the nonlinear analysis of reinforced concrete flexural slab system," thesis presented to University College

of Swansea, at Swansea, U.K., in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
Bodner, S. R., and Partom, Y. (1975). "Constitutive equations for elastic-viscoplastic strain-hardening materials." *J. Appl. Mech.*, ASME, 42(2), 385–389.
Buyukozturk, O. (1977). "Nonlinear analysis of reinforced concrete structure." *Comput. Struct.*, 7(1), 149–156.
Chen, A. C. T., and Chen, W. F. (1975). "Constitutive relations for concrete." *J. Engrg. Mech. Div.*, ASCE, 101(4), 465–481.
Chen, E. Y.-T. (1981). "Numerical simulation of reinforced concrete subjected to multiaxial stress conditions," thesis presented to the University of Illinois, at Urbana-Champaign, Urbana-Champaign, Ill., in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
Chen, W. F. (1982). *Plasticity in reinforced concrete*. McGraw-Hill, New York, N.Y.
Chen, W. F., and Ting, E. C. (1980). "Constitutive models for concrete structures." *J. Engrg. Mech. Div.*, ASCE, 106(1), 1–19.
Darwin, D., and Pecknold, D. A. (1974). "Inelastic model for cyclic biaxial loading for reinforced concrete." *Civil Engineering Studies, Structural Research Series No. 409*, Univ. of Illinois at Urbana-Champaign, Urbana-Champaign, Ill.
Drucker, D. C. (1950). "Some implications of work hardening and ideal plasticity." *Q. Appl. Math.*, 7(4), 411–418.
Drucker, D. C. (1951). "A more fundamental approach to plastic stress-strain relations." *Proc., First U.S. National Congress of Applied Mechanics*, ASME, New York, N.Y., 487–491.
Elwi, A. A., and Murray, D. W. (1979). "A 3D hypoelastic concrete constitutive relationship." *J. Engrg. Mech. Div.*, ASCE, 105(4), 623–641.
Hand, F. R., Pecknold, D. A., and Schnobirch, W. C. (1972). "A layered finite element nonlinear analysis of reinforced concrete plates and shells." *Civil Engineering Studies, Structural Research Series, No. 389*, Univ. of Illinois at Urbana-Champaign, Urbana-Champaign, Ill.
Hen, D. J., and Chen, W. F. (1987). "Constitutive modeling in analysis of concrete structures." *J. Engrg. Mech.*, ASCE, 113(4), 577–593.
Hu, H.-T., and Schnobrich, W. C. (1988). "Nonlinear analysis of plane stress state reinforced concrete under short term monotonic loading." *Civil Engineering Studies, Structural Research Series, No. 539*, Univ. of Illinois at Urbana-Champaign, Urbana-Champaign, Ill.
Kupfer, H., Hilsdorf, H. K., and Rusch, H. (1969). "Behavior of concrete under biaxial stresses." *ACI J.*, 66(8), 656–666.
Lade, P. V., Nelson, R. B., and Ito, Y. M. (1987). "Nonassociated flow and stability of granular materials." *J. Engrg. Mech.*, ASCE, 113(9), 1302–1318.
Li, H.-B., et al. (1986). "A new method for the coupling of finite element and boundary element discretized subdomains of elastic bodies." *Comput. Methods Appl. Mech. Engrg.*, 54(2), 161–185.
Mendelson, A. (1983). *Plasticity: Theory and application*. Reprint Ed., Robert E. Krieger Publishing Co., Malabar, Fla.
Mikkola, M. J., and Schnobrich, W. C. (1970). "Material behavior characteristics for reinforced concrete shells stressed beyond the elastic range." *Civil Engineering Studies, Structural Research Series, No. 367*, Univ. of Illinois at Urbana-Champaign, Urbana-Champaign, Ill.
Mroz, Z. (1963). "Non-associated flow laws in plasticity." *Journal de Mecanique*, 2(1), 21–42.
Murray, D. W., et al. (1979). "Concrete plasticity theory for biaxial stress analysis." *J. Engrg. Mech. Div.*, ASCE, 105(6), 989–1006.
Saenz, L. P. (1964). Discussion of "Equation the stress-strain curve of concrete," by P. Desayi and S. Krishnan, *ACI J.*, 61(9), 1229–1235.
Schnobrich, W. C., and Hu, H.-T. (1985). "Use of finite element techniques to predict nonlinear behavior of reinforced concrete structures." *Finite elements in computational mechanics*, I, Bombay, India, Pergamon Press, 395–412.
Sinha, B. P., Gerstle, K. H., and Tulin, L. G. (1964). "Stress-strain relations for

concrete under cyclic loading." *ACI J.*, 61(2), 195–211.
Vermeer, P. A., and de Borst, R. (1984). "Non-associated plasticity for soils, concrete and rock." *Heron*, Delft Univ., The Netherlands, 29(3), 1–64.

Appendix II. Notation

The following symbols are used in this paper:

$[C], [C]_s$ = unsymmetric and symmetrized material constitutive matrices;
$[C]_e, [C]_p, [C]_{ep}$ = elastic, plastic, and elastic-plastic material matrices;
c_1, c_2, c_3 = variables associated with yield functions;
d = differentiational operator;
E_c, E_t = initial modulus and tangent modulus for concrete;
$F(\ ), G(\ )$ = loading function;
$f(\ )$ = yield function;
f_i = nodal forces;
f'_c = maximum compressive strength of concrete;
f'_t = maximum tensile strength of concrete;
$g(\ )$ = plastic potential function;
H = plastic hardening modulus;
q = variable associated with equivalent uniaxial stress-strain curve;
$u_i, \underline{u}_i$ = nodal displacements, nodal virtual displacements;
W, W_p = total work and plastic work;
α, β = material constants;
δ = variational operator;
ϵ_f = maximum strain on equivalent uniaxial stress-strain curve;
ϵ_o = strain corresponding to f'_c in uniaxial compression test;
ϵ_* = strain corresponding to f'_c on equivalent uniaxial stress strain curve;
$\underline{\epsilon}, \underline{\epsilon}_e, \underline{\epsilon}_p$ = equivalent total strain, equivalent elastic strain, and equivalent plastic strain;
$\{\epsilon\}, \{\underline{\epsilon}\}$ = vectors of total strains and virtual strains;
$\{\epsilon\}_p, \{\epsilon\}_p$ = vectors of elastic and plastic strains;
λ = scalar factor;
ν = Poisson's ratio;
Π = potential energy;
$\underline{\sigma}$ = equivalent stress;
$\{\sigma\}$ = vector of total stresses;
σ_f = stress corresponding to ϵ_f on equivalent uniaxial stress strain curve;
σ_m = mean stress;
σ_1, σ_2 = principal stresses;
τ_{oct} = octahedral shear stress;
$\{\ \}$ = row vector;
$\langle\ \rangle = \{\ \}^T$ = column vector; and
$[\ ]$ = matrix.